Connectedness, non superstability, and generic elements

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The aim of this lecture is to prove that $\mathbb{F}\omega$ is not superstable but it is connected.

1 Some defintions

- **Left generic**: If X is a definable subset of G then we say X is left generic in G if finitely many left translates of X by elements of G cover G

$$\exists h_1, ..., h_r : G = \bigcup_i h_i X$$

- **Generic formula:** We say that a formula is generic if the set it determine is generic.
- **Generic over**: G is a stable group. If $g \in G$and A is a set of parameters from G then we say that G is generic over A or tp(g/A) is generic if every formula in tp(g/A) is generic in G.
- **Type generic**: We say that a type is generic if it contains only generic formula.
- **Algebraic:** We say that $g \in G$ is algebraic over $A \subseteq G$ if there exists X definable set over A (with a formula that its parameters are in A) such that $g \in X$ and X is finite.
- **Connected:** G is said to be connected if G has no proper definable subgroup of finite index.

2 Superstability

We know that Zlil Sela prooved that $\mathbb{F}\omega$ is stable. His demonstration is over our comprehension .So instead to understand it, we will try to show that $\mathbb{F}\omega$ is not superstable.

Lemma 0.1: *Suppose X is a left generic set then X contains all but finitely many elements of the basis.*

Lemma: *$X \subseteq G^n$ definable set by $\phi(x_1, x_2, ..., x_n, A)$. So if $\sigma \in Aut_A(G)$ ($\Leftrightarrow \forall a \in A, \sigma(a) = a$) then $\sigma(X) = X$.*

This means that if $(g_1, ..., g_n)$ satisfy ϕ so $\sigma(g_1), ..., \sigma(g_n)$ satisfy ϕ .

Proof of the lemma 0.1:

$\mathbb{F}\omega$ is a free group $\mathbb{F}\omega =< e_1, e_2, ... >$ when e_i is a basis element.

X definable subset of $\mathbb{F}\omega$ by $\phi(x_1, ..., x_l, a_1, ..., a_k)$.

Let N be such that the parameters in the formula defining X as well as $h_1, ..., h_s$ are words in $e_1, ..., e_N$ and their inverses.

X is generic this means $\mathbb{F}\omega = \bigcup_{i=1}^{s} h_i X$.

Now, $e_{N+1} \in h_i X$ then $h_i^{-1} e_{N+1} \in X$. So we want to proof that $\exists \sigma \in Aut_{<e,e_N>}(\mathbb{F}\omega)$ such that $\sigma(h_i^{-1} e_{N+1}) = e_{N+1}$ and

$\forall j \geq N+1$ $\exists \sigma_j \in Aut_{<e,e_N>}(\mathbb{F}\omega)$ such that $\sigma_j(h_i^{-1} e_{N+1}) = e_j$and we will finish.

See the lecture 1 about Automorphism.

There exists $f \in Aut_{\mathbb{F}\omega}(\mathbb{F}\omega)$ such that

$$e_{N+1} \longmapsto h_i^{-1} e_{N+1}$$

$$e_j \longmapsto e_j$$

for every $j \neq N+1$.

Take his inverse σ :

$$h_i^{-1} e_{N+1} \longmapsto e_{N+1}$$

$$e_j \longmapsto e_j$$

for every $j \neq N+1$.

Now we define

$$\Gamma_j : e_{N+1} \longmapsto e_j$$
$$e_j \longmapsto e_{N+1}$$
$$e_k \longmapsto e_k$$

for all $j \neq k \neq N+1$.

Let $\sigma_j = \Gamma_j \circ \sigma$.

All these this homomrphisms are Automorphisms.

$\Longrightarrow$Then we find

$$\sigma(h_i^{-1}e_{N+1}) = e_{N+1} \in X$$

and

$$\sigma_j(h_i^{-1}e_{N+1}) = e_j \in X$$

and we finish.

Proposition 0.2: *$G = \mathbb{F}\omega$ is not superstable.*

Before to start to prove this lemma there are some points to understand:

1) $g, h \in \mathbb{F}\omega \; if \; g^2 = h^2 \Longrightarrow g = h$

2) If G is stable so there exist at least 1-type generique.

3) **Theorem of Poizat**: *G superstable. If $A = \{\boldsymbol{a}\}$ and g is generic and algebric over A then **a** is generic.*

Proof of proposition 0.2:

Assume by contradiction that $\mathbb{F}\omega$ is superstable.

Take X define by $\phi(x) : \exists y \; x = y^2$.

There are no basis element in X (because a basis element cannot be a square). X is not generic according to lemma 0.1.

$\mathbb{F}\omega$ is superstable and in particular stable. Thus there exist a $g \in \mathbb{F}\omega$ that is generic. $g^2 \in \phi$ so $g^2 = \boldsymbol{a}$ is not generic.

We want to show that **a** is algebric:

We have a formula :

$$\varphi(x, a) \; : x^2 = a$$
$$\varphi(g, a) \; : g^2 = a$$

so g is contain in the formula and we have a finite set (because there are a unique root in the free group).

Thus, according to Poizat **a** is generic. Absurdity!

Proposition 0.4: *$\mathbb{F}\omega$ is connected.*

Proof:

Suppose by contradiction that $\mathbb{F}\omega$ is not connected.

Suppose that X is a definable finite index subgroup. Then X and all its translates $h_iX$are all left generic definable sets.

$\Rightarrow$each of them contain infinitely many of $e_n's$ contadicting their disjoinctness.

Proposition 0.5:e_{n+1} *is generic over* $e_1, ..., e_n$. (this means that e_{n+1} realize definable sets with parameters $e_1, ..., e_n$)

Proof:

By the above e_{n+1} is contained in all the generic sets defined over $e_1, ..., e_n$.

$\mathbb{F}_\mathrm{n}$is stable so there is a 1-type generic (over any parameters) and e_{n+1}must realize it.