1 Lecture

Definition 1. A countable group G is said to be homogeneous if for every two tuples $g_1, \ldots, g_k, h_1, \ldots, h_k \in G$ s.t. $\operatorname{tp}^G(g_1, \ldots, g_k) = \operatorname{tp}^G(h_1, \ldots, h_k)$ there is an automorphism $\sigma \in \operatorname{Aut}(G)$ s.t. $\sigma(g_i) = h_i$ for $1 \leq i \leq k$.

The opposite direction is true. Let $g_1, \ldots, g_k, h_1, \ldots, h_k \in G$ be tuples and let $\varphi \in \operatorname{Aut}(G)$ be an automorphism s.t. $\varphi(g_i) = h_i$ for all i then $\operatorname{tp}^G(g_1, \ldots, g_k) = \operatorname{tp}^G(h_1, \ldots, h_k)$. We notice that $g \sim h \iff \operatorname{tp}(g) = \operatorname{tp}(h)$ is an equivalence relation. In the general case the orbits of the automorphism group are a subpartition of the partition induced by the relation of being in the same type. If G is homogeneous then these partition coincide.

Proposition 2. *Let $S = \{s_1, \ldots, s_k\}$ be a finite set with $k \geq 2$. Let $g_1, \ldots, g_n, h_1, \ldots, h_n \in F_S$ s.t. $tp^{F_S}(g_1, \ldots, g_n) = tp^{F_S}(h_1, \ldots, h_n)$ there is an homomorphism $\nu : F_S \to F_S$ s.t. $\nu(g_i) = h_i$ for $1 \leq i \leq k$.*

Proof. For every $g \in F_s$ we construct the term $w_g(x_1, \ldots, x_k)$ by taking the letters $s_1, \ldots, s_k \in F_S$ and replacing them by variables $x_1, \ldots, x_k$. For example let $g = s_1 s_2 s_1^{-1} s_2^{-1} \in F_S$ so $w_g(x_1, x_2) = x_1 x_2 x_1^{-1} x_2^{-1}$ and $w_g(s_1, s_2) = g$. Let

$$\varphi(z_1, \ldots, z_n) = \exists x_1, \ldots, x_k \bigwedge_{i=1}^{n} w_{g_i}(x_1, \ldots, x_k) = z_i$$

clearly $F_S \vDash \varphi(g_1, \ldots, g_n)$ because for $s_1, \ldots, s_k$ we have $w_{g_i}(s_1, \ldots, s_k) = g_i$. Because $g_1, \ldots, g_n$and $h_1, \ldots, h_n$ are of the same type $F_S \vDash \varphi(h_1, \ldots, h_n)$ meaning there are $t_1, \ldots, t_k \in F_s$ s.t. $w_{g_i}(t_1, \ldots, t_k) = h_i$. Let

$$\begin{aligned} \nu : F_S &\to F_S \\ s_i &\mapsto t_i \end{aligned}$$

and we get $\nu(g_i) = h_i$. □

Definition 3. For every $g \in F_S$ we define

$$\varphi'_g(z, x_1, \ldots, x_k) = (w_g(x_1, \ldots, x_k) = z)$$

Remark 4. We notice that it is enough to discuss existence of an homomorphism for a 1 tuple we can achieve the general case by conjunction of formulas of the form$\varphi'_g(z, x_1, \ldots, x_k)$.

Let $g, h \in F_S$ with $\operatorname{tp}^{F_S}(g) = \operatorname{tp}^{F_S}(h)$.

Idea. Naively we want to construct a formula $\psi^k(y_1,\ldots,y_k)$ s.t. for all $t_1,\ldots,t_k\in F_S$ we will have

$$F_S\vDash\psi(t_1,\ldots,t_k)\iff t_1,\ldots,t_k\text{ are the images of }s_1,\ldots,s_k\text{under an automorphism}$$

Then we will have $F_S\vDash\varphi'_g(g,s_1,\ldots,s_k)\wedge\psi(s_1,\ldots,s_k)$ and for every $t_1,\ldots,t_k\in F_S$ s.t. $F_S\vDash\varphi'_g(h,t_1,\ldots,t_k)\wedge\psi(t_1,\ldots,t_k)$ the map σ sending $s_i\mapsto t_i$ is an automorphism with $\sigma(g)=h$.

Problem. The formula $\exists x_1,\ldots,x_k\forall z\,\psi^k(x_1,\ldots,x_k)\wedge\neg\psi^k(x_1,\ldots,x_k,z)$ would contradict Sela's Theorem that all non abelian free groups have the same first order theory.

But Sela has another theorem that might come in handy

Theorem 5. *Free groups have "relative co-Hopf property" . Let $g\in F_S$ be an element not contained in any proper free factor and let $\sigma:F_S\to F_S$ be an injective homomorphism s.t. $\sigma(g)=g$. Then σ is also surjective.*

Idea. Construct a formula $\psi^k(y_1,\ldots,y_k)$ s.t. for all $t_1,\ldots,t_k\in F_S$ we will have

$$F_S\vDash\psi(t_1,\ldots,t_k)\iff t_1,\ldots,t_k\text{ are the images of }s_1,\ldots,s_k\text{under an inj. homomorphsim}$$

Then we will get an inj. homomorphism $\sigma(g)=h$. Symmetrically we can get an inj. homomorphism $\tau(h)=g$. From the relative co Hopf property $\tau\circ\sigma$ is an automorphism so also σ is an automorphism.

This idea works for $k=2$. This proof that F_2 to is homogeneous is due to Nies.

Proposition 6. *Let $\sigma:F_S\to F_S$ be an homomorphism let $K=\langle s_1,\ldots,s_l\rangle$ be a free factor. Then $\sigma|_K$ is injective iff $\langle\sigma(s_1),\ldots,\sigma(s_l)\rangle$ is a subgroup of rank l.*

Proof. Nielsen-Schreier +Hopf property. □

Proposition 7. *Let $t_1,t_2\in F_S$ then $\langle t_1,t_2\rangle$ is of rank 2 iff $[t_1,t_2]\neq 1$*

Proof. $\langle t_1,t_2\rangle$ is of rank less then 2 iff $\langle t_1,t_2\rangle$ is abelian (Nielsen-Schreier) □

So the formula ψ^2 is

$$\psi^2(x_1,x_2)=([x_1,x_2]\neq 1)$$

Fact 8. *Let $J\leq F_S$ be a free factor of F_S and let $K\leq F_S$ be a subgroup. Then the subgroup $J\cap K$ is a free factor of K.*

Claim 9. Let F_{S_1} and F_{S_2} be free groups on the sets S_1, S_2 let $g \in F_{S_1}, h \in F_{S_2}$ be elements and $\nu : F_{S_2} \to F_{S_1}$ and injective homomorphism s.t. $\nu(h) = g$. If F_{S_1} has a free factor J_1 s.t. $\text{rank} J = 1$ and $g \in J$ then F_{S_2} has a free factor J_2 with $\text{rank} J_2 = 1$ and $h \in J_2$.

Proof. Let $K \leq F_{S_1}$ be the group $K = \text{Im}\nu$. Because of fact 8 the subgroup $J_1 \cap K$ is a free factor of K. We also know that $J_1 \cap K \leq J_1$ and a subgroup of free group of rank 1 is a free group of rank 1. Then $J_2 = \nu^{-1}(J_1 \cap K)$ satisfies the requirements. We notice that $\nu|_{J_2} : J_2 \to J_1 \cap K$ is an isomorphism. □

Theorem 10. *Let S be a set $|S| = 2$ an let $g, h \in F_S$ be elements s.t. $tp^{F_2}(g) = tp^{F_2}(h)$ then there is an automorphism $\sigma \in Aut(F_S)$ with $\varphi(g) = h$*

Proof. The formula $\exists x_1 x_2 \, \varphi_g(z, x_1, x_2) \wedge [x_1, x_2] \neq 1$ gives us an injective homomorphism $\nu_1 : F_S \to F_S$ with $\nu_1(g) = h$ the formula $\exists x_1 x_2 \, \varphi_h(z, x_1, x_2) \wedge [x_1, x_2] \neq 1$ gives us an injective homomorphism $\nu_2 : F_S \to F_S$ s.t. $\nu_2(h) = g$. If F_S is indecomposable over g then $\nu_2 \circ \nu_1$ is an automorphism because of the relative co-Hopf property. If F_S isn't indecomposable over g then from Claim 9 there are free factors $J_1, J_2 \leq F_S$ s.t. $\text{rank}(J_1), \text{rank}(J_2) = 1$, $g \in J_1, h \in J_2$ and $\nu_2(J_2) = J_1$. Then we can extend $\nu_2|_{J_2} : J_2 \to J_1$ to an automorphism $\sigma : F_S \to F_S$ with $\sigma|_{J_2} = \nu_1|_{J_2}$ and $\sigma(h) = g$. □

Problem. There can't be appropriate formulas ψ^k for $k > 2$ because the sentence $\exists x_1, x_2 \forall x_3, \ldots, x_k \psi^2(x_1, x_2) \wedge \neg\psi^k(x_1, \ldots, x_k)$ would contradict Sela.

Fact 11. *Let $H < F_S$ be a subgroup. Then H has infinite index iff there exists a subgroup $H < K < F_S$ of infinite index s.t. H is a free factor of K.*

So the sentence $\exists x_1, x_2 \forall x_3, \ldots, x_k \psi^2(x_1, x_2) \wedge \neg\psi^k(x_1, \ldots, x_k)$ can be interpreted as meaning F_S has a finite index subgroup of rank 2. The Nielsen Schreier formula for the rank of a finite index subgroup states $r = 1 + i(k - 1)$ with i being the index of the subgroup. The formula can hold only if $k = 2$ and $i = 1$. Meaning the only possible case of a finite index subgroup of rank 2 is $F_2 \leq F_2$.

Theorem 12. *Let $g, h \in F_S$ s.t. g is not contained in a proper free factor of F_S. There are a finite set of proper quotients $\eta_i : F_S \to Q_i$ s.t. for every non injective homomorphism $\sigma : F_S \to F_S$ s.t. $\sigma(g) = h$ there exists an automorphism $\tau \in Mod_{\langle g \rangle}(F_S)$ s.t. $\sigma \circ \tau$ factors through one of the quotient.*

Idea. Define a relation $\zeta(x_1,\dots,x_k,y_1,\dots,y_k)$s.t. for all $t_1,\dots,t_k,t'_1,\dots,t'_k \in F_S$

$$F_S \vDash \zeta(t_1,\dots,t_k,t'_1,\dots,t'_k)$$
$$\Updownarrow$$
$$\text{There exists } \tau \in \mathrm{Mod}_{\langle g\rangle}(F_S) \text{ and } \sigma : F_S \to F_S \text{ s.t. } \sigma(g)=h$$
$$\text{and } \sigma(s_i)=t_i \text{ and } \sigma\circ\tau(s_i)=\tau(t'_i)$$

So again this is impossible but, finally, we can construct a formula ζ s.t

$$F_S \vDash \zeta(t_1,\dots,t_k,t'_1,\dots,t'_k)$$
$$\Uparrow$$
$$\text{There exists } \tau \in \mathrm{Mod}_{\langle g\rangle}(F_S) \text{ and } \sigma : F_S \to F_S \text{ s.t. } \sigma(g)=h$$
$$\text{and } \sigma(s_i)=t_i \text{ and } \sigma\circ\tau(s_i)=\tau(t'_i)$$

Let $u_i \in F_S$ be words s.t. $u_i \in \ker(\eta_i)$. We construct

$$\psi_{g,h}(x_1,\dots,x_k) = \forall y_1,\dots,y_k \zeta(x_1,\dots,x_k,y_1,\dots,y_k) \to \bigwedge_i \neg\varphi'_{u_i}(1,y_1,\dots,y_k)$$

Let σ be an homomorphism and $t_1,\dots,t_k \in F_S$ s.t. $\sigma(s_i)=t_i$. If $F_S \vDash \psi_{g,h}(t_1,\dots,t_k)$ we can conclude σ is injective because of Theorem 12.
We are left with two assignments

1. Construct ζ.
2. Prove that $F_S \vDash \varphi'_g(g,s_1,\dots,s_k) \wedge \psi_{g,h}(s_1,\dots,s_k)$

We won't prove 1. There is a JSJ decomposition of F_S relative to $\langle g\rangle < F_S$. We analyze the way in which Dehn twists act on the vertex groups of the JSJ decomposition. For regular vertex groups the Dehn twist conjugate by an element of F_S. Some of the amalgamation products come from splitting a surface along a simple curve. In this case the Dehn twist act as an automorphism of the surface group. If the amalgamation product doesn't come from splitting a surface the Dehn twist conjugates the surface group. Let $\tau \in \mathrm{Mod}_{\langle g\rangle}(F_S)$ be a modular automorphism and $K \le F_S$ is regular vertex subgroup then there is a $f \in F_S$ s.t. $\tau(k) = fkf^{-1}$for every $k \in K$. Let $H \le F_S$ be a surface vertex subgroup then $\tau(H) = fHf^{-1}$ (f is different) but it doesn't necessarily conjugate every single elements. Let $\sigma : F_S \to F_S$ be a homomorphism then for $k \in K$ we have $\sigma\circ\tau(k) = \sigma(fkf^{-1}) = \sigma(f)\sigma(k)\sigma(f)^{-1}$. For H we have that subgroup $\sigma\circ\tau(H)$ is conjugate to the subgroup$\sigma(H)$. So $\sigma(H)$ is abelian iff $\sigma\circ\tau(H)$ is abelian. Each vertex in the JSJ decomposition gives us a set of generators. The edges of the JSJ decomposition give us relations between the generators. Now we can write $F_S \cong \langle a_1, \ldots, a_k | r_1, \ldots, r_j \rangle$. The relation can be expressed as a set of equations $\Sigma(x_1, \ldots, x_k) = \bigwedge_{i=1}^{j} \varphi'_{r_i}(1, x_1, \ldots, x_k)$. We construct the formulas

$$\begin{aligned} \zeta_{RVertex}(x_1, \ldots, x_l, y_1, \ldots, y_l) &= \exists z \bigwedge_i z x_i z^{-1} = y_i \\ \zeta_{SVertex}(x_1, \ldots, x_l, y_1, \ldots, y_l) &= \left(\bigwedge_{i,j=1}^{l} [x_i, x_j] = 1 \right) \longleftrightarrow \left(\bigwedge_{i,j=1}^{l} [y_i, y_j] = 1 \right) \end{aligned}$$

Let $a_1, \ldots, a_l$ a subset of generators from a regular vertex group and let $t_1, \ldots, t_l, t'_1, \ldots, t'_l \in F_S$ be elements s.t. $\sigma(a_i) = t_i$ and $\sigma \circ \tau(a_i) = t'_i$. Then $F_S \vDash \zeta_{RVertex}(t_1, \ldots, t_l, t'_1, \ldots, t'_l)$. Let $b_1, \ldots, b_l$ be a subset of generators from a surface group and let $t_1, \ldots, t_l, t'_1, \ldots, t'_l \in F_S$ be elements s.t. $\sigma(b_i) = t_i$ and $\sigma \circ \tau(b_i) = t'_i$ then $F_S \vDash \zeta_{SVertex}(t_1, \ldots, t_l, t'_1, \ldots, t'_l)$. For each vertex on the JSJ decomposition we construct one of these two formulas so

$$\zeta(x_1, \ldots, x_k, y_1, \ldots, y_k) = \bigwedge_i \zeta_{RVertex}(x_1, \ldots, x_{l_i}, y_1, \ldots, y_{l_i}) \bigwedge_i \zeta_{SVertex}(x_1, \ldots, x_{l_i}, y_1, \ldots, y_{l_i})$$

To put it all together. We write g as a word w_g in the generators from the JSJ decomposition so the final formuls is $\varphi'_g(z, x_1, \ldots, x_l) \wedge \Sigma(x_1, \ldots, x_l) \wedge \psi_{gh}(x_1, \ldots, x_l)$.