Ampleness of the free group - First order theory of the free group seminar notes

Leor Neuhauser

March 19, 2019

Def 1: a stable theory T is n-ample if (after adding some parameters) there are in some large enough model $a_1, .., a_n$ s.t.

1. a_0 forks with $a_n$over $\emptyset$ (really over the added parameters)
2. a_{i+1}does not fork with $a_0, .., a_{i-1}$over a_i for $1 \leq i < n$
3. $acl^{eq}(a_0) \cap acl^{eq}(a_1) = acl^{eq}(\emptyset)$
4. $acl^{eq}(a_0, ..., a_{i-1}, a_i) \cap acl^{eq}(a_0, ..., a_{i-1}, a_{i+1}) = acl^{eq}(a_0, ..., a_{i-1})$ for $1 \leq i < n$

The "eq" means that we work in an expanded structure M^{eq}, that we'll define (somewhat informally) shortly.

Def 2: a tuple c is a canonocal parameter of a definable (with parameters) set $D \subset M^n$ if $\phi(M, c) = D$ and for $c' \neq c$ $\phi(M, c') \neq D$. If every definable set has a canonical parameter we say the theory has elimination of imaginaries (EI).

Fact 1: If every equivalence class of every $\emptyset$-definable equivalence relation on M^n has a canonical parameter then the theory eliminates imaginaries.

Fact 2: if a is a canonical parameter of D, then b is also a canonical parameters of D iff a, b are inter-definable $a \in dcl(b)$, $b \in dcl(a)$.

We want every definable set to have a canonical parameter, it is enough to add a canonical parameter for every equivalence relation.

Def 3: We'll buid M^{eq} as an expansion of M. For every $\emptyset$-definable equivalence relation E on M^n, add as **elements** the equivalnce classes of E. Those new elements are called imaginaries. Also add to the language the projections $\pi_E : M^n \to M^n/E$.

Fact 3: M^{eq} eliminiates imaginaries

Cor 1: M has elimination of imagenries iff every imaginary is inter-definable with a real tuple.

Def 4: acl^{eq}, dcl^{eq} are acl, dcl taken in M^{eq}.

Def 5: M has weak EI if for every imaginary $e \in M^{eq}$ there is real tuple $c \in M$ s.t. $e \in dcl^{eq}(c)$ and $c \in acl^{eq}(e)$.

Note: for our purposes, we only look at acl^{eq},so weak EI is sufficent.

We want to expand the theory of free groups with only mild equivalence relations so that it would have weak EI

Def 6: Let $\mathbb{F}$ be a non abelian free group. the following equivalence relations are called basic:

- (conjugation) aE_1b iff $\exists g : g^{-1}ag = b$
- (m-left-coset) $(a_1, b_1)E_2^m(a_2, b_2)$ iff $b_1 = b_2 = 1$ or else $C(b_1) = C(b_2) = \langle b \rangle$ and $a_1^{-1}a_2 \in \langle b^m \rangle$.
- (m-right-coset) $(a_1, b_1)E_3^m(a_2, b_2)$ iff $b_1 = b_2 = 1$ or else $C(b_1) = C(b_2) = \langle b \rangle$ and $a_1a_2^{-1} \in \langle b^m \rangle$.
- (m,n-bicoset) $(a_1, b_1, c_1)E_4^{m.n}(a_2, b_2, c_2)$ iff $a_1 = a_2 = 1$ or $c_1 = c_2 = 1$ or else $C(a_1) = C(a_2) = \langle a \rangle$, $C(c_1) = C(c_2) = \langle c \rangle$ and $\exists g \in \langle a^m \rangle, h \in \langle c^n \rangle$: $gb_1h = b_2$.

Thm: Denote $\mathbb{F}^{we}$ the expansion of $\mathbb{F}$ by the equivalence classes of basic equivalence relations. $\mathbb{F}^{we}$ has weak EI.

Thm: For every n, the theory of the free group is n-ample.

Proof: Work in $\mathbb{F} = \mathbb{F}_{2n+3} = \langle e_1, .., e_{2n+3} \rangle$ over the parameter e_1, e_2 and build $a_0, .., a_n$ by

$$a_0 = e_3$$

$$a_1 = a_0[e_4, e_5] = e_3[e_4, e_5]$$

$$\vdots$$

$$a_n = a_{n-1}[e_{2n+2}, e_{2n+3}] = e_3[e_4, e_5]...[e_{2n+2}, e_{2n+3}]$$

Reminder: We say that an element g is generic over parameters A if every formula in $tp(g/A)$ is genric, where a definable set is generic if it has finitely many left translates that cover the whole group. In the free group, an element is generic if it satisfies the unique generic type. As we've seen in Rachel's lecture (proposition 0.5 in here notes) e_{i+1} is generic over $e_1, .., e_i$.We will use a strengthend version of this.

Thm (Pillay): $b_1, .., b_k$ can be extended to a basis iff they are independent (do not fork with each other) and generic.

Prop 1: a_0 forks with $a_n$over e_1, e_2.

Proof: $a_0 = e_3$ is generic over e_1, e_2, and $e_1, e_2, e_4, ..., e_{2n+2}, e_{2n+3}, a_n$ is a basis of $\mathbb{F}_{2n+3}$ so by Pillay a_n is generic over e_1, e_2. If we would have also a_0 d.n.f.w. a_n over e_1, e_2, so by Pillay we would get that e_1, e_2, a_0, a_n could be extended to a basis, $e_1, e_2, e_3, a_n, b_1, ...b_{2n-1}$. In particular in the abelization $\mathbb{F}/[\mathbb{F}, \mathbb{F}] \simeq \mathbb{Z}^{2n+3}$ and the image under the quoitent of the basis $\bar{e_1}, \bar{e_2}, \bar{e_3}, \bar{a_n}, \bar{b_1}, ...b_{2n-1}$ is a generating set. But

$$\bar{a_n} = \bar{e_3}\overline{[e_4, e_5]}...\overline{[e_{2n+2}, e_{2n+3}]} = \bar{e_3}$$

contradiction.

Prop 2: a_{i+1}does not fork with $a_0, .., a_{i-1}$over a_i, e_1, e_2 for $1 \leq i < n$

Proof: By Pillay e_{2i+4}, e_{2i+5} dnf with $e_1, ..., e_{2i+3}$. we can switch any side of this relation with things from it's algebraic closure to get e_{2i+4}, e_{2i+5} dnf with $e_1, e_2, a_0, ..., a_i$ so by lowering things to parameters e_{2i+4}, e_{2i+5} dnf$/e_1, e_2, a_i$ with $a_0, ..., a_{i-1}$, so by raising some parameters to the right $e_{2i+4}, e_{2i+5}a_i$ dnf$/e_1, e_2, a_i$ with $a_0, ..., a_{i-1}$, and lastly by the fact $a_{i+1} = a_i[e_{2i+4}, e_{2i+5}]$ so we get a_{i+1} dnf$/e_1, e_2, a_i$ with $a_0, ..., a_{i-1}$, as needed.

Prop 3: $acl^{eq}(e_1, e_2, a_0) \cap acl^{eq}(e_1, e_2, a_1) = acl^{eq}(e_1, e_2)$

Lemma: $acl(e_1, e_2, a_i) = \langle e_1, e_2, a_i \rangle$, $(i = 0, 1)$

Pf Lemma: e_1, e_2, a_i can be extended to a basis,

$$e_1, e_2, a_i, e_4, ..., e_{2n+3}$$

for $i = 0$ $a_0 = e_3$ so it is clearly a basis. For $i = 1$ $a_1 = e_3[e_4, e_5]$ so $e_3 \in \langle a_1, e_4, e_5 \rangle$, thus those $2n + 3$ elements generate all $\mathbb{F}$,so they are a basis. in praticular $\langle e_1, e_2, a_i \rangle$ is a free factor of , so $acl(e_1, e_2, a_i) \subset \langle e_1, e_2, a_i \rangle$. the other direction is clear.

Pf Prop 3: For real elements,

$$acl(e_1, e_2, a_0) \cap acl(e_1, e_2, a_1) = \langle e_1, e_2, a_0 \rangle \cap \langle e_1, e_2, a_1 \rangle \supset \langle e_1, e_2 \rangle$$

$\langle e_1, e_2, a_0 \rangle \cap \langle e_1, e_2, a_i \rangle$ is a free factor of order either 2 or 3. If it's order was 3, then it would mean $\langle e_1, e_2, a_0 \rangle = \langle e_1, e_2, a_1 \rangle$, but it can't be because e_4, e_5 don't show in $\langle e_1, e_2, e_3 \rangle$. Thus the order is 2 and $\langle e_1, e_2, a_0 \rangle \cap \langle e_1, e_2, a_1 \rangle = \langle e_1, e_2 \rangle = acl(e_1, e_2)$.

For imaginary elements, we only look at acl so weak EI is enough, so we only need to check the basic equivalence classes. (Free factors are elementary substructures, so the acl^{eq}in the big structure is the same as the acl^{eq} in the small structure).

- For conjugation, suppose $e \in \mathbb{F}/E_1$ s.t. $e \in acl^{eq}(e_1, e_2, a_0) \cap acl^{eq}(e_1, e_2, a_1)$. $e = [g]_{E_1} = [h]_{E_1}$ for $g \in \langle e_1, e_2, a_0 \rangle$, $h \in \langle e_1, e_2, a_1 \rangle$. We can assume WLOG g, h are in cyclically reduced forms

$$g = w_1(e_1, e_2)a_0^{k_1}w_2(e_1, e_2)a_0^{k_2}...w_r(e_1, e_2)a_0^{k_r}$$

$$h = v_1(e_1, e_2)a_1^{l_1}v_2(e_1, e_2)a_1^{l_2}...v_s(e_1, e_2)a_1^{ls}$$

  but g, h are congugates, so their cyc reduced forms should be cyclic permutation of each other. In praticular $h \in \langle e_1, e_2, a_0 \rangle$, so $h \in \langle e_1, e_2 \rangle$. thus $e = [h]_{E_1} \in acl^{eq}(e_1, e_2)$.

- For m-left-coset, suppose $e \in \mathbb{F}/E_2^m$ s.t. $e \in acl^{eq}(e_1, e_2, a_0) \cap acl^{eq}(e_1, e_2, a_1)$. $e = [g_1, g_2]_{E_2^m} = [h_1, h_2]_{E_2^m}$ for $g_i \in \langle e_1, e_2, a_0\rangle$, $h_i \in \langle e_1, e_2, a_1\rangle$. We have $(g_1, g_2)E_2^m(h_1, h_2)$, i.e. either $g_2 = h_2 = 1$, and then $(g_1, 1)E_2^m(1, 1) \in \langle e_1, e_2\rangle$, or else $C(g_2) = C(h_2) = \langle b\rangle$ and $g_1^{-1}h_1 \in \langle b^m\rangle$. $C(g_2) \subset \langle e_1, e_2, a_0\rangle$, $C(h_2) \subset \langle e_1, e_2, a_1\rangle$, so $b \in \langle e_1, e_2\rangle$. $g_2, h_2 \in \langle b\rangle \subset \langle e_1, e_2\rangle$ and $g_1^{-1}h_1 \in \langle b^m\rangle \subset \langle e_1, e_2\rangle$ so $h_1 \in \langle e_1, e_2, a_0\rangle \cdot \langle e_1, e_2\rangle \subset \langle e_1, e_2\rangle$. $h_i \in \langle e_1, e_2\rangle$, so
$$e = [h_1, h_2]_{E_2^m} \in acl^{eq}(e_1, e_2)$$
- the same for right and double cosets.

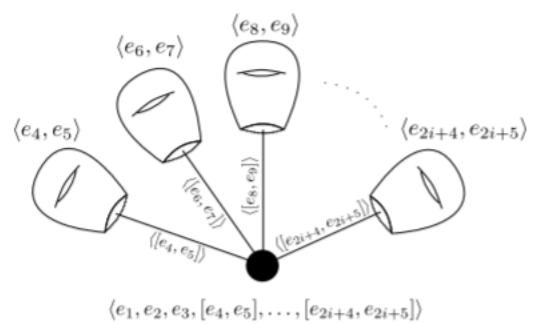
Prop 4: $acl^{eq}(e_1, e_2, a_0, ..., a_{i-1}, a_i) \cap acl^{eq}(e_1, e_2, a_0, ..., a_{i-1}, a_{i+1}) = acl^{eq}(e_1, e_2, a_0, ..., a_{i-1})$ for $1 \leq i < n$

Lemma 1: Suppose $\gamma \in acl^{eq}(e_1, e_2, a_0, ..., a_{i-1}, a_i)$;

- if γ is real $\gamma \in \langle e_1, e_2, a_0, ..., a_{i-1}, a_i\rangle =: A$
- if $\gamma = [c]_E$ for a basic E then $\exists d \in A$ s.t. $\gamma = [d]_E$ (i.e. $[c]_E \cap A \neq \emptyset$).

Note: We can't use the "trick" with free factor because A is not necessarily a free factor.

Proof: Notice that $A = \langle e_1, e_2, e_3, [e_4, e_5], ..., [e_{2i+2}, e_{2i+3}]\rangle$ look at the graph of groups

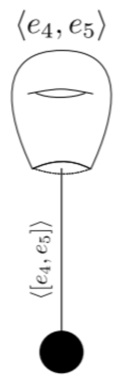


We have

$$\mathbb{F}_{2i+3} = (((A *_{[e_{2i+2}, e_{2i+3}]} \langle e_{2i+2}, e_{2i+3}\rangle) *_{[e_{2i}, e_{2i+1}]} \langle e_{2i}, e_{2i+1}\rangle)...) *_{[e_4, e_5]} \langle e_4, e_5\rangle$$

If we have a JSJ decombosition with one of the vertices cointaining A, then that vertice is $acl(A)$. By the universal property we can collapse the JSJ to get the above graph. In this graph A is a vertice, so it must be that $A = acl(A)$ proving the first point. for imaginaries,

if $\gamma = [c]_{E_1}$: Write $A_0 = A$, $A_{l+1} = A_l *_{[e_{2(i-l)+2}, e_{2(i-l)+3}]} \langle e_{2(i-l)+2}, e_{2(i-l)+3}\rangle$. Suppose $l \leq i$ is the minimal s.t. $[c]_{E_1} \cap A_l \neq \emptyset$. Asumme by contadiction $l > 0$, WLOG just assume $l = i$. We can assume (up to conjugation) that c is in cyclically reduced form with respect to $A_{i-1} *_{[e_4,e_5]} \langle e_4, e_5\rangle$, $c = c_1 \cdot ... \cdot c_m$. By the minimality assumption $c_i \notin A_{i-1} = \langle e_1, e_2, e_3, [e_4, e_5], e_6, ..., e_{2i+3}\rangle$. for some $j \leq m$ $c_j \notin A_{i-1}$.s



To get a contradiction, we would like to show that $[c]_{E_1}$ has infinitley many orbits under $Aut_A(\mathbb{F}_{2i+3})$, i.e. the orbit of c has infinitley many conjugacey classes. we would have liked to use dehn twists, but the orbit under dehn twists may have only finitely many conj classes. So we use a more complicated homeomorphism of the surface, a pseudo-Anosov homeomorphism h. h can be extended to an automorphism of $\mathbb{F}_{2i+3}$ by preserving A_{i-1}, in praticular preserving A. We have $\{[h^k(c)]_{E_1} | k < \omega\}$ is finite. .there is an infinite $I \subset \omega$ s.t. c and $h^k(c)$ are conjugates for any $k \in I$.

- If $m = 1$, then $c = c_1 \in \langle e_4, e_5\rangle \backslash \langle [e_4, e_5]\rangle$ $\langle e_4, e_5\rangle$ is a free factor so for every $k \in I$ c and $h^k(c)$ must be conjugates in $\langle e_4, e_5\rangle$, in contadiction to one of the properties of pseudo-Anosov (if $I \subset \omega$ is infinite, $\{h^k(c) | k \in I\}$ has infinite conjugation classes)

- If $m > 1$, then $\forall k \in I$ $h^k(c) = h^k(c_1) \cdot ... \cdot h^k(c_m)$ is conjugate to $c_1 \cdot ... \cdot c_m$, which is in cyclicly reduced form, so $h^k(c)$ is obtained from c by a cyclic permutation and then conjugation from the boundary, $h^k(c) = b_k^{-1} c_{p_k(1)} ... c_{p_k(m)} b_k$ for p_k a cyc permutation and $b \in \langle [e_4, e_5]\rangle$. this contradicts another property of pseudo-Anosov.

For the other basic eq classes the proof goes in similar ways.

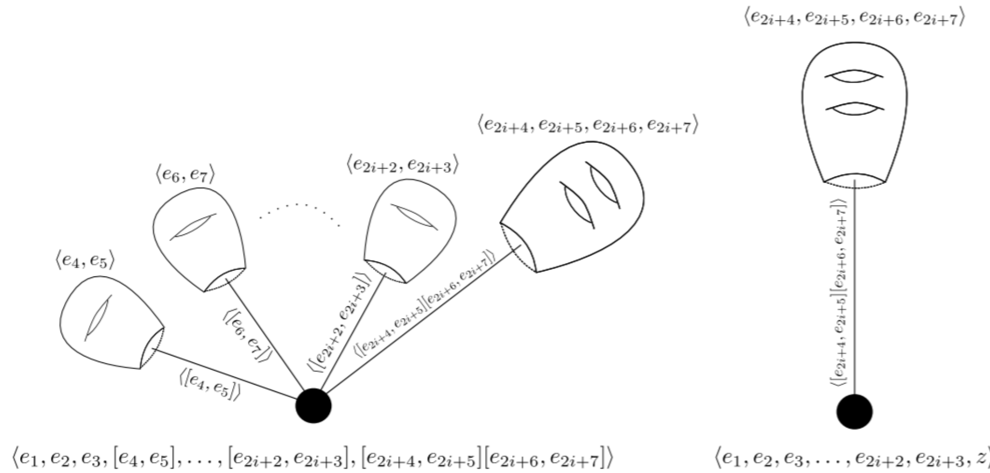
Remark: Lemma 1 aplies also to $acl^{eq}(e_1, e_2, a_0, ..., a_{i-1}, a_i)$.

Lemma 2: Suppose $\gamma \in acl^{eq}(e_1, e_2, a_0, ..., a_{i-1}, a_{i+1})$;

- if γ is real $\gamma \in \langle e_1, e_2, a_0, ..., a_{i-1}, a_{i+1}\rangle =: A$

- if $\gamma = [c]_E$ for a basic E then $\exists d \in A$ s.t. $\gamma = [d]_E$.

Pf Lemma 2: The proof goes the same, except we need to look at the graphs of groups



Pf Prop 4: Denote $A = \langle e_1, e_2, a_0, ..., a_{i-1}, a_i\rangle = \langle e_1, e_2, e_3, [e_4, e_5], ..., [e_{2i+2}, e_{2i+3}]\rangle$
$B = \langle e_1, e_2, a_0, ..., a_{i-1}, a_{i+1}\rangle = \langle e_1, e_2, e_3, [e_4, e_5], ..., [e_{2i}, e_{2i+1}], [e_{2i+2}, e_{2i+3}][e_{2i+4}, e_{2i+5}]\rangle$

Suppose $\gamma \in acl^{eq}(A) \cap acl^{eq}(B)$. If γ is real, $\gamma \in A \cap B$. $\gamma \in \mathbb{F}_{2i+5} = \langle e_1, e_2, .., e_{2i+1}\rangle * \langle e_{2i+2}, .., e_{2i+5}\rangle$, we can write γ in a normal form with respect to this splitting $\gamma = c_1 b_1 ... c_m b_m$. $c_j \in \langle e_1, e_2, .., e_{2i+1}\rangle, b_j \in \langle e_{2i+2}, .., e_{2i+5}\rangle$. Because $\gamma \in A \cap B$ we have $b_j \in \langle [e_{2i+2}, e_{2i+3}]\rangle \cap \langle [e_{2i+2}, e_{2i+3}][e_{2i+4}, e_{2i+5}]\rangle$, so the b_j must be trivial and $\gamma \in \langle e_1, e_2, .., e_{2i+1}\rangle \cap A \cap B \subset \langle e_1, e_2, e_3, [e_4, e_5], ..., [e_{2i}, e_{2i+1}]\rangle$ as needed.

If γ is a E_1 equivalence class, by lemma 1+2 we can write $\gamma = [c]_{E_1} = [d]_{E_1}$ for $c \in A$, $d \in B$. We can assume that c, d are in cyc reduced form w.r.t the splitting $\langle e_1, e_2, .., e_{2i+1}\rangle * \langle e_{2i+2}, .., e_{2i+5}\rangle$ c and d must be cyclic permutations of each other, so like before they must both live in $\langle e_1, e_2, .., e_{2i+1}\rangle$, thus they must both live in $\langle e_1, e_2, e_3, [e_4, e_5], ..., [e_{2i}, e_{2i+1}]\rangle$ so $\gamma \in acl^{eq}(e_1, e_2, e_3, [e_4, e_5], ..., [e_{2i}, e_{2i+1}])$

similarly if γ is a E_2^m equivalence class, by lemma 1+2 we can write $\gamma = [c_1, c_2]_E = [d_1, d_2]$ for $c_1, c_2 \in A$, $d_1, d_2 \in B$. if $c_2 = d_2 = 1$ it is obvious, else $C(c_2) = C(d_2) = \langle b\rangle$, and $c_1^{-1} d_1 \in \langle b^m\rangle$. It must be that $b \in A \cap B = \langle e_1, e_2, e_3, [e_4, e_5], ..., [e_{2i}, e_{2i+1}]\rangle$ so $d_1, d_2 \in \langle e_1, e_2, e_3, [e_4, e_5], ..., [e_{2i}, e_{2i+1}]\rangle$ as needed.