SOME LOWER BOUNDS ON SHELAH RANK IN THE FREE GROUP - LECTURE NOTES

We will give equivalent defenition of foking, with notion of definable sets.

Lemma. *(Alternative Defenition Of Forking) A formula ϕ forks over A iff there is a a sequense $\{\sigma_n\} \subset Aut_A(\mathbb{M})$ and an integer $k < \omega$ such that $\sigma_n(X)$ (where X is the set defined by ϕ) is k-wise disjoint. Thus we can say thay "The defineable set forks over A" instead talking about the formula defining X.*

There is an importent lemma for evaluating Shelah Rank

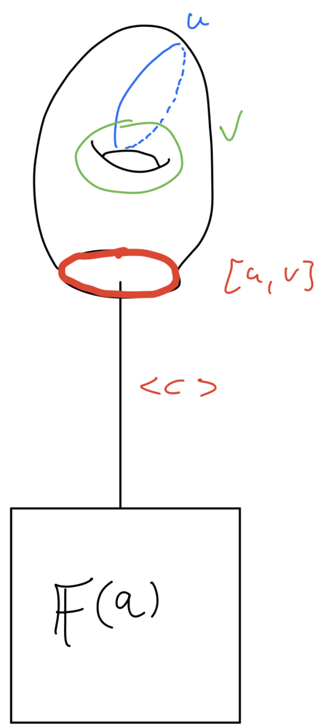
Lemma. *Let $\mathbb{M}$ be a model with a fixed parameter subset A. Suppose that $X \subset M^d$ is definable (not necessarily over A only). If There exist a subset Y of X, definable over $\mathbb{M}$, an integer k, ans a sequence $\sigma_n \in Aut_A(\mathbb{M})$ such that the translates $\sigma_n(Y)$ are k-wise disjoint, and the σ_n preserve X set-wise, then*

$$R_A^{\infty}(X) \geq R_A^{\infty}(Y) + 1$$

[Talk about graph of proups, floor and structures]

Let T be a tower structure with one floor of the graph of groups Γ

FIGURE 0.1. *



We want to show that

$$R^{\infty}_{\mathbb{F}(a,b)}\left(V^{T}\right)\geq\omega$$

Recall that

$$T=\left\langle a,b,u,v\mid\overbrace{[a,b]=[u,v]}^{\Sigma_T}\right\rangle$$

Fact. *T is a model of the theory of the free groups, and it's an elementery extention of $\mathbb{F}(a,b)$.*

So we may look at V^T as $V^T=\left\{(x,y)\in T^2\mid[a,b]=[x,y]\right\}$

In order to see that $R^{\infty}_{\mathbb{F}(a,b)}\left(V^{T}\right)>\omega$ we will show that for every $n<\omega$ there is X_n such that $R^{\infty}_{\mathbb{F}(a,b)}\left(X_n\right)\geq n$, and we will get that

$$R^{\infty}_{\mathbb{F}(a,b)}\left(V^{T}\right)\geq R^{\infty}_{\mathbb{F}(a,b)}\left(X_n\right)\geq n$$

for every n so

$$R^{\infty}_{\mathbb{F}(a,b)}\left(V^{T}\right)\geq\omega$$

In order to see it we will construct, for every $n<\omega$ a set X_n where $R^{\infty}_{\mathbb{F}(a,b)}\left(X_n\right)\geq n$.

Definition. Define $\tau_a, \tau_b \in \mathrm{Aut}_{\mathbb{F}(a,b)}(T)$ by

$$\tau_u = \begin{array}{c} a \mapsto a \\ b \mapsto b \\ u \mapsto u \\ v \mapsto vu \end{array}, \; \tau_v = \begin{array}{c} a \mapsto a \\ b \mapsto b \\ u \mapsto uv \\ v \mapsto v \end{array}$$

Fact. *τ_u^2 and τ_v^2 form a base for a free sup-group of $Aut_{\mathbb{F}(a,b)}(T)$ of rank 2.*

Prooving that $R^{\infty}_{\mathbb{F}(a,b)}\left(V^T\right) \geq \omega$

Define

- For $n = 0$ $X_0 = \emptyset$, of course $R^{\infty}_{\mathbb{F}(a,b)}(X_0) \geq 0$.
- For $n = 1$ we define $X_1 = \{(u, v)\}$. If we take $\sigma_n = I_T$ we get $\sigma_n(X_0) = \emptyset$ are disjoint (they are empty), and $\sigma_n(X_1) = X_1$ so $R^{\infty}_{\mathbb{F}(a,b)}(X_1) \geq R^{\infty}_{\mathbb{F}(a,b)}(X_0) + 1 \geq 1$
- For $n = 2$ we define

  $$X_2 = \left\{\left(u, uv^{2k}\right) \mid k \in \mathbb{Z}\right\}$$

  X_2 is definable by the formula

  $$\varphi(x, y) \sim x = u \wedge \exists z \left([z, u] = 1 \wedge y = uz^2\right)$$

  For every $k \in \mathbb{Z}$ $\tau_u^{2k}(X_1) = \left\{\left(u, vu^{2k}\right)\right\}$ and they form a disjoint sequence $\left\{\tau_a^{2k}(X_1)\right\}$ and τ_a^{2k} preserve X_2 set-wise, so

  $$R^{\infty}_{\mathbb{F}(a,b)}(X_2) \geq R^{\infty}_{\mathbb{F}(a,b)}(X_1) + 1 \geq 2$$

- For $n = 3$ we notice that $X_2 = \left\langle\tau_u^2\right\rangle \cdot (u, v)$ and we define

  $$\begin{aligned} X_3 &= \left\langle\tau_u^2\right\rangle \cdot \tau_v^2 \cdot \left\langle\tau_u^2\right\rangle \cdot X_1 = \left\langle\tau_u^2\right\rangle \cdot \left(\tau_v^2 [X_2]\right) = \\ &= \bigcup_{k \in \mathbb{Z}} \bigcup_{m \in \mathbb{Z}} \left\{\tau_u^{2k} \tau_v^2 \tau_u^{2m}(u, v)\right\} \end{aligned}$$

  If this union is not disjoint we get that there is $k_1 m_1 k_2, m_2$ such that

  $$\tau_u^{2k_1} \tau_v^2 \tau_u^{2m_1}(u, v) = \tau_u^{2k_2} \tau_v^2 \tau_u^{2m_2}(u, v)$$

  this means that $\tau_u^{2k_1} \tau_v^2 \tau_u^{2m_1}$ and $\tau_u^{2k_2} \tau_v^2 \tau_u^{2m_2}$ identify on u and v, and so on a and b, so they identify on the generators so they identical. $\left\langle\tau_u^2, \tau_v^2\right\rangle$ is free group so $k_1 = k_2, m_1 = m_2$ and X_3 is disjoint union. Furthermore $\tau_u^{2k}(X_3) = X_3$ so

  $$R^{\infty}_{\mathbb{F}(a,b)}(X_3) \geq R^{\infty}_{\mathbb{F}(a,b)}(X_2) + 1 \geq 3$$

- For each n, after we defined X_n we define

$$\begin{aligned} X_{n+1} &= \left\langle\tau_u^2\right\rangle \cdot \tau_v^2(X_n) = \\ &= \left\{w\left(\tau_u^2, \tau_v^2\right)(u, v) \mid w \text{ is a word in } x, y \text{ that starts and ends with } x \text{ and has no powers of } y \text{ and has } y \; n \text{ -} \right. \\ &= \bigcup_{k \in \mathbb{Z}} \tau_u^{2k} \circ \tau_v^2(X_n) \end{aligned}$$

  If this union is not disjoint then there are two words that start with some power of τ_u^2 and agree on the generators of T, so this they that same so the first power of τ_u^2 is the same, so they are in the same set int the union.

X_{n+1} is preserved by τ_u^{2k} $(k \in \mathbb{Z})$. This means that $R^{\infty}_{\mathbb{F}(a,b)}(X_{n+1}) \geq R^{\infty}_{\mathbb{F}(a,b)}(X_n) + 1 \geq n+1$.

For Toruos With Genuse Bigger then 1

We will give general idea about the prove for the case of the two hole toruse, and hopefully the reader will see understand the general idea for some genuse.

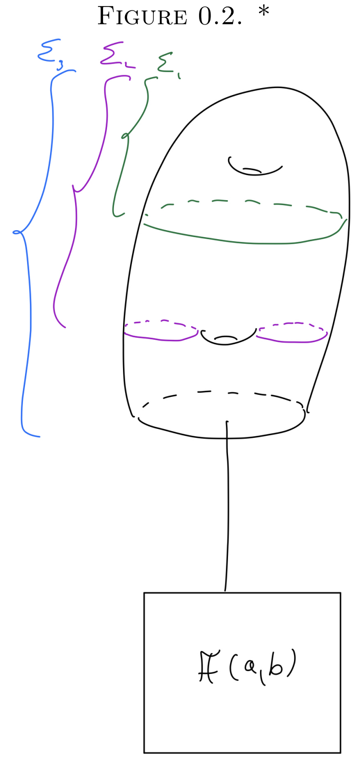
A theorem that will be with out proof will help us to do the same trick as before.

Theorem. *If $\Sigma_0 \subset \Sigma_1$ are surfaces with bounderies where $\pi_1(\Sigma_0)$ is proper subgroup of $\pi_1(\Sigma_1)$ then there is a $\varphi \in Aut(\Sigma_1)$ such that*

$$\langle Aut(\Sigma_0), \varphi \rangle = Aut(\Sigma_0) * \langle \varphi \rangle$$

If we divide the double toruse to three surfaces like so

Figure 0.2. *



In this case we get

$$V^T = \left\{ (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in T^3 \mid \Sigma_T() \right\}$$

Where $\Sigma_T()$ are the relation defining T.

$$Y_1 = \{ (y_1, z_1, \bar{x}_2, \bar{x}_3) \mid \Sigma_T(), [y_1, z_1] = [u_1, v_1] \}$$

$$Y_2 = \{ (y_1, z_1, q_2, p_2, \bar{x}_3) \mid \Sigma_T(), [y_1, z_1] = [u_1, v_1], q_2 p_2 = [y_1, z_1] \}$$

We can think about the groups like subgorups $\mathrm{Aut}_{\partial\Sigma_0}(\Sigma_0) \leq \mathrm{Aut}_{'\partial\Sigma_1}(\Sigma_1) \leq \mathrm{Aut}_{\mathbb{F}(a,b)}(T)$