# Non-Equationality of The Free Group

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## 1 Intro and Motivation

### 1.1 Outline

In this talk I will prove that the theory of $\mathbb{F}$ is non-equational. The talk is based on an article by Isabel Müller and Rizos Sklinos, building on previous work of Sela. (https://arxiv.org/abs/1703.04169)

The talk will consists of:

1. Definitions, motivation and combinatorial tools
2. Proof $\mathbb{F}$ is non-equational
3. (If time permits)
   Proof $G_1 * G_2$ (excluding $\mathbb{Z}_2 * \mathbb{Z}_2$) is non-equational, using Bass-Serre Theory.

**Note**: Most of what's written down was not in the talk itself, but I tried to prove as much of the nontrivial and semi-trivial statements as I could. Some of these are my own proofs so there may be some inaccuracies.

### 1.2 Definitions

**Definition 1.1.** Let $T$ be a first order theory. A formula $\varphi(x,y)$ is called an *equation* in $x$ ($x$ is a variable, $y$ is a parameter, both are tuples) if any collection of instances $\varphi(x,b)$ is equivalent to a finite sub-collection in $T$. That is, for any $(b_i)_{i\in I}$ we have a finite $I_0 \subseteq I$ s.t.:

$$\bigcap_{i\in I}\varphi(x,b_i) = \bigcap_{i\in I_0}\varphi(x,b_i)$$

Equivalently,

*Claim* 1.2. $\varphi(x,y)$ is an equation in $x$ iff the family of intersections of instances $\varphi(x,b)$ has the DCC.

*Proof.* (Same proof as for "Module is Noetherian iff sub-modules are finitely generated")

($\Rightarrow$) If we have an ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of indexing sets that induces a descending chain:

$$\bigcap_{i\in I_1}\varphi(x,b_i) \supseteq \bigcap_{i\in I_2}\varphi(x,b_i) \supseteq \cdots$$

Then if we take $I = \bigcup_{n\in\mathbb{N}} I_n$ we have $\bigcap_{i\in I}\varphi(x,b_i) = \bigcap_{i\in I_0}\varphi(x,b_i)$ for a finite subset $I_0$ of $I$ and so there must be some $I_{n_0}$ for which $I_0 \subseteq I_{n_0}$. So for any $n > n_0$, $I_0 \subseteq I_m \subseteq I_n$ and:

$$\bigcap_{i\in I_n}\varphi(x,b_i) \supseteq \bigcap_{i\in I}\varphi(x,b_i) = \bigcap_{i\in I_0}\varphi(x,b_i) \supseteq \bigcap_{i\in I_n}\varphi(x,b_i)$$

therefore $\bigcap_{i\in I_n}\varphi(x,b_i) = \bigcap_{i\in I}\varphi(x,b_i)$ and the chain is stable from $n_0$.

($\Leftarrow$) If we have the DCC and there is an indexing set $I$ for which we take $\bigcap_{i\in I}\varphi(x,b_i)$, then take the family

$$\Omega = \left\{\bigcap_{i\in J}\varphi(x,b_i)\,|J \subseteq I \text{is finite}\right\}$$

It is nonempty since $|M| = \bigcap_{i\in\varnothing} \varphi(x, b_i) \in \Omega$ and any descending chain has a lower bound in this set. (A descending chain $\left(\bigcap_{i\in J_n} \varphi(x, b_i)\right)$ is stable hence there is some $n_0$ s.t. the chain equals $\bigcap_{i\in J_{n_0}} \varphi(x, b_i) \in \Omega$ from $n_0$ and this is its lower bound). Therefore (Zorn) it has a minimal element $\bigcap_{i\in I_0} \varphi(x, b_i)$. Now, it is always true that:

$$\bigcap_{i\in I_0} \varphi(x, b_i) \supseteq \bigcap_{i\in I} \varphi(x, b_i)$$

Since $I_0 \subseteq I$ so if these are not equal there exists $j \in I \setminus I_0$ such that:

$$\bigcap_{i\in I_0} \varphi(x, b_i) \supsetneqq \bigcap_{i\in I_0\cup\{j\}} \varphi(x, b_i) \supseteq \bigcap_{i\in I} \varphi(x, b_i)$$

So $I_0 \cup \{j\} \in \Omega$ contradicts the minimality of $I_0$. Hence there is an equality. □

*Remark* 1.3. Equationality is a generalization of Noetherianity of Modules and Rings. Collections of $b_i$ are the restricting conditions (Such as generators of ideals in a ring) and they have an ACC on their closure. $\bigcap_i \varphi(x, b_i)$ are the underlying sets (algebraic/closed sets of the Zariski topology) and they admit a DCC (Noetherian topological space). Moreover, a ring is Noetherian iff every ideal is finitely generated, corresponding to our first definition.

**Definition 1.4.** A theory $T$ is *$n$-equational* if every formula $\varphi(x, y)$ where $|x| = n$ ($x$ is an $n$-tuple) is a Boolean combination of equations.

**Definition 1.5.** $T$ is *equational* if it is $n$-equational for all $n \in \mathbb{N}$.

**Example 1.6.** Some examples of equations:

1. $x = y$
2. For any definable equivalence relation $\sim$, $x \sim y$ is an equation.
3. $x \neq y$ is not an equation (for an infinite model)
   →So for an equation $\varphi$; $\neg\varphi$ is not necessarily an equation.
4. In algebraically closed fields $\varphi(x, y) \leftrightharpoons \sum_\alpha f_\alpha(y)\, x^\alpha = 0$ is an equation.
   →Precisely because $\Bbbk[x]$ is a Noetherian ring (Hilbert's Basissatz)

## 1.3 Some properties of equations

*Remark* 1.7. $\varphi(x, y)$ is **not** an equation iff there exists an infinite sequence $\{c_n\}_{n\in\mathbb{N}}$ and the following is a properly decreasing chain:

$$\varphi(x, c_0) \supsetneqq \varphi(x, c_0) \cap \varphi(x, c_1) \supsetneqq \cdots \supsetneqq \bigcap_{k\leq n} \varphi(x, c_k) \supsetneqq \cdots$$

*Proof.* (⇐) Immediate from the DCC.

(⇒) If $\varphi(x, y)$ is not an equation, then there is a set $I$ such that $\bigcap_{i\in I} \varphi(x, b_i)$ but for every finite subset $J \subseteq I$ it holds that:

$$\bigcap_{i\in I} \varphi(x, b_i) \supsetneqq \bigcap_{i\in J} \varphi(x, b_i)$$

We can then choose $\{b_{i_n}\}_{n\in\mathbb{N}}$ and indexing sets $I_n = \{i_k\}_{k=0}^n$ such that:

$$\bigcap_{i\in I_0} \varphi(x, b_i) \supsetneqq \bigcap_{i\in I_1} \varphi(x, b_i) \supsetneqq \cdots \supsetneqq \bigcap_{i\in I_n} \varphi(x, b_i) \supsetneqq \cdots$$
$$\varphi(x, b_{i_0}) \supsetneqq \varphi(x, b_{i_0}) \cap \varphi(x, b_{i_1}) \supsetneqq \cdots \supsetneqq \bigcap_{k\leq n} \varphi(x, b_{i_k}) \supsetneqq \cdots$$

Since all $I_n$ are indeed finite. Set $c_n = b_{i_n}$ and we are done. □

**Fact 1.8.** *If for arbitrarily large $n$ there exists a sequence $\{b_i\}_{i<n}$ such that the following is a decreasing sequence:*

$$\varphi(x,b_0) \supsetneqq \varphi(x,b_0) \cap \varphi(x,b_1) \supsetneqq \cdots \supsetneqq \bigcap_{i<n} \varphi(x,b_i)$$

*Then there exists an infinite properly decreasing sequence:*

$$\varphi(x,c_0) \supsetneqq \varphi(x,c_0) \cap \varphi(x,c_1) \supsetneqq \cdots \supsetneqq \bigcap_{0\leq k\leq n} \varphi(x,c_k) \supseteq \cdots$$

The proof of this fact uses a compactness argument (which we didn't discuss in the seminar) on formulas of the type:

$$\bigwedge_{i=1}^{n} (\exists x)\left(\varphi(x,y_i) \wedge \left(\neg \bigwedge_{j=0}^{i-1} \varphi(x,y_j)\right)\right)$$

One can also follow the proof of the stronger argument in the proof of Proposition 2.11 here:
https://www.math.uwaterloo.ca/∼rmoosa/ohara.pdf

*Remark* 1.9. For $\varphi(x;y)$, $\varphi^{op}(x;y) \coloneqq \varphi(y;x)$ is an equation (w.r.t. $y$).

*Proof.* Suffices to show one direction from symmetry. Assume $\varphi(x,y)$ is not an equation in $y$. Then from remark 1.7 We can then choose $\{a_n\}_{n\in\mathbb{N}}$ such that:

$$\varphi(a_0,y) \supsetneqq \varphi(a_0,y) \cap \varphi(a_1,y) \supsetneqq \cdots \supsetneqq \bigcap_{k\leq n} \varphi(a_k,y) \supsetneqq \cdots$$

Then there exist for each $j \in \mathbb{N}$;

$$b_j \in \bigcap_{i<j} \varphi(a_j,y) \setminus \varphi(a_j,y) =$$

whence $\models \varphi(a_i,b_j)$ for $i<j$ but $\not\models \varphi(a_i,b_i)$.

This condition precisely means that for each $n$:

$$\varphi(x,b_n) \supsetneqq \varphi(x,b_n) \cap \varphi(x,b_{n-1}) \supsetneqq \cdots \supsetneqq \bigcap_{0\leq k\leq n} \varphi(x,b_k)$$

And by compactness there exists a sequence $\{c_k\}_{k\in\mathbb{N}}$:

$$\varphi(x,c_0) \supsetneqq \varphi(x,c_0) \cap \varphi(x,c_1) \supsetneqq \cdots \supsetneqq \bigcap_{0\leq k\leq n} \varphi(x,c_k) \supseteq \cdots$$

Contradicting $\varphi$ being an equation in $x$. □

**Lemma 1.10.** *Finite conjunctions and disjunctions of equations are equations:*

*Proof.* Suffices to show for two equations. Assume $\varphi_1(x,y), \varphi_2(x,y)$ are equations. Then take the formulas: for some indexing sets $I$ there exist $I_1, I_2$ such that:

$$\bigcap_{i\in I} \varphi_1(x,b_i) = \bigcap_{i\in I_1} \varphi_1(x,b_i)$$
$$\bigcap_{i\in I} \varphi_2(x,b_i) = \bigcap_{i\in I_2} \varphi_2(x,b_i)$$

And we have:

$$\begin{aligned}\bigcap_{i\in I}\left(\varphi_1(x,b_i)\vee\varphi_2(x,b_i)\right) &= \left(\bigcap_{i\in I}\varphi_1(x,b_i)\right)\cup\left(\bigcap_{i\in I}\varphi_2(x,b_i)\right) = \\ &= \left(\bigcap_{i\in I_1}\varphi_1(x,b_i)\right)\cup\left(\bigcap_{i\in I_2}\varphi_2(x,b_i)\right) = \\ &= \left(\bigcap_{i\in I_1\cup I_2}\varphi_1(x,b_i)\right)\cup\left(\bigcap_{i\in I_1\cup I_2}\varphi_2(x,b_i)\right) = \\ &= \bigcap_{i\in I_1\cup I_2}\left(\varphi_1(x,b_i)\vee\varphi_2(x,b_i)\right)\end{aligned}$$

Noting that adding $I_2$ to $I_1$ does not restrict $\bigcap_{i\in I_1}\varphi_1(x,b_i)$ further since it already includes all the conditions of $\varphi_1(x,b_i)$ for $i\in I$ and vice versa on $\varphi_2(x,b_i)$. And similarly:

$$\begin{aligned}\bigcap_{i\in I}\left(\varphi_1(x,b_i)\wedge\varphi_2(x,b_i)\right) &= \left(\bigcap_{i\in I}\varphi_1(x,b_i)\right)\cap\left(\bigcap_{i\in I}\varphi_2(x,b_i)\right) = \\ &= \left(\bigcap_{i\in I_1}\varphi_1(x,b_i)\right)\cap\left(\bigcap_{i\in I_2}\varphi_2(x,b_i)\right) = \\ &= \left(\bigcap_{i\in I_1\cup I_2}\varphi_1(x,b_i)\right)\cap\left(\bigcap_{i\in I_1\cup I_2}\varphi_2(x,b_i)\right) = \\ &= \bigcap_{i\in I_1\cup I_2}\left(\varphi_1(x,b_i)\wedge\varphi_2(x,b_i)\right)\end{aligned}$$

Where since $I_1, I_2$ are finite, their union is finite. □

**Corollary 1.11.** *Finite conjunctions and disjunctions of co-equations are co-equations:*

*Proof.* Again, suffices to show for two co-equations. Note that for $\varphi,\psi$ equations,

$$\neg\varphi(x,y)\wedge\neg\psi(x,y)=\neg\left(\varphi(x,y)\vee\psi(x,y)\right)$$
$$\neg\varphi(x,y)\vee\neg\psi(x,y)=\neg\left(\varphi(x,y)\wedge\psi(x,y)\right)$$

The rest follows from the lemma. □

### 1.4 Motivation

Question: Does 1-equational imply equational? No contradictions yet. Open problem. We will show $\mathbb{F}$ is not 1-equational.

Motivation: Collect as many examples of non-equational theories. $\mathbb{F}$ is an example of a stable but non-equational theory which is a surprising result.

## 2 Tools

We try to approach the concept of equationality in a combinatorial manner, starting with a combinatorial criterion for some formula being an equation:

**Lemma 2.1.** *$\varphi(x,y)$ is* ***not*** *an equation iff and only if for arbitrarily large $n\in\mathbb{N}$ there are $n$-tuples $(a_i),(b_i)$ such that $\models\varphi(a_i,b_j)$ for $i<j$ but $\not\models\varphi(a_i,b_i)$.*

*Proof.* We will show this criterion for $\varphi(x,y)$ not being an equation in $y$. It not being an equation in $x$ follows from remark 1.9.

($\Leftarrow$) If such tuples exist then

$$b_j\in\bigcap_{i<j}\varphi(a_i,y)\setminus\bigcap_{i\leq j}\varphi(a_i,y)$$

And then:

$$\varphi(a_0,y) \supsetneqq (\varphi(a_0,y)\cap\varphi(a_1,y)) \supsetneqq \cdots \supsetneqq \bigcap_{k\le n}\varphi(a_k,y)$$

And we have an arbitrarily long descending chain, hence there must be an infinite unstable descending chain. So $\varphi(x,y)$ is not an equation in $y$ and therefore not an equation in $x$.

($\Rightarrow$) Conversely, if $\varphi(x,y)$ is not an equation in $y$ then there exists an infinite series $\{a_i\}$ and an infinite properly descending chain:

$$\varphi(a_0,y) \supsetneqq (\varphi(a_0,y)\cap\varphi(a_1,y)) \supsetneqq \cdots \supsetneqq \bigcap_{k\le n}\varphi(a_k,y)$$

So we can find $b_j \in \bigcap_{i<j}\varphi(a_i,y) \setminus \bigcap_{i\le j}\varphi(a_i,y)$ for all $j\in\mathbb{N}$. In particular if we fix $n$ we can take the tuples $(a_i)_{i\le n}$ and $(b_i)_{i\le n}$ and our criterion is satisfied. □

**Fact 2.2.** *Any Boolean combination $\varphi$ of atomic formulas $(\varphi_k)$ is equivalent to a formula in disjunctive normal form (DNF)*

$$\psi \coloneqq \bigvee_{n\le m}\left(\bigwedge_{j\le \ell_n}\psi_{i,j}\right)$$

*Where $\psi_{i,j}$ is some $\varphi_{k_{i,j}}$ or $\neg\varphi_{k_{i,j}}$*

**Corollary 2.3.** *Assume $\varphi(x,y)$ equivalent to Boolean combination of equations. Then $\varphi(x,y)$ is equivalent to a formula of the form:*

$$\psi(x,y) \coloneqq \bigvee_{n\le m}\left(\psi_1^n(x,y)\wedge\neg\psi_2^n(x,y)\right)$$

*For some equations $\psi_1^i,\psi_2^i$ and $m\in\mathbb{N}$.*

*Proof.* Write $\psi(x,y)$ in DNF. Inside each element of the disjunction, there is a finite conjunction of equations and co-equations. Each such element is equivalent to a conjunction of an equation and a co-equation therefore $\varphi(x,y)$ is equivalent to $\psi(x,y)$. □

**Lemma 2.4.** *If $\varphi(x,y)$ is a formula then if for arbitrarily large $n\in\mathbb{N}$ exist $n\times n$ matrices*

$$A_n \coloneqq (a_{ij})$$
$$B_n \coloneqq (b_{ij})$$

*such that $\vDash \varphi(a_{ij},b_{kl})$ iff $i\neq k$ or $(i,j)=(k,l)$ then $\varphi(x,y)$ is* not *equivalent to a formula of the form $\psi_1(x,y)\wedge\neg\psi_2(x,y)$ where $\psi_1$ and $\psi_2$ are equations.*

*Proof.* **Part 1:** First we prove that for arbitrarily large $n$ every row (in both matrices simultaneously) witnesses that $\neg\varphi(x,y)$ is not equivalent to an equation.

Fix $i_0$. Then we have $(a_{i_0j})$ and $(b_{i_0j})$ as $n$-tuples with:

$$(\vDash \varphi(a_{i_0j},b_{i_0l})) \iff j=l$$
$$(\vDash \neg\varphi(a_{i_0j},b_{i_0l})) \iff j\neq l$$

Specifically if $j<l$, $\vDash \neg\varphi(a_{i_0j},b_{i_0l})$. Therefore $\varphi$ is not equivalent to an equation by Lemma 2.1

**Part 2:** Assume the contrary of the conclusion, i.e. that $\varphi(x,y)\equiv\psi_1(x,y)\wedge\neg\psi_2(x,y)$ and reach a contradiction.

Assuming this, we have that $\neg\varphi(x,y)\equiv\neg\psi_1(x,y)\vee\psi_2(x,y)$.

If for some $i_0$ we have $\vDash \psi_1(a_{i_0j},b_{i_0,l})$ for all $j,l$ then for $i_0$ we have

$$\neg\varphi(a_{i_0j},b_{i_0,l}) \leftrightarrow \psi_2(a_{i_0j},b_{i_0l})$$

For all $j,l$. Contradicting that $\psi_2$ is an equation but by Part 1 of the proof the LHS satisfies the criterion for not being an equation.

**Part 3:** Now from part 2, for any $i$ there exist some $j_i, l_i$ with $\vDash \neg\psi_1\left(a_{ij_i}, b_{il_i}\right)$. Set new $n$-tuples

$$\left(a_{ij_i}\right), \left(b_{kl_k}\right)$$

For $i \neq k$ we have $\vDash \varphi\left(a_{ij_i}, b_{kl_k}\right)$ (from the condition of the lemma). Remembering our original assumption $\varphi(x,y) \equiv \psi_1(x,y) \wedge \neg\psi_2(x,y)$, For $i < k$ then we have $\vDash \varphi\left(a_{ij_i}, b_{kl_k}\right)$ and hence $\vDash \psi_1\left(a_{ij_i}, b_{kl_k}\right)$ and also $\nvDash \psi_1\left(a_{ij_i}, b_{il_k}\right)$. Contradicting that $\psi_1$ is an equation again by the criterion of Lemma 2.1. □

**Proposition 2.5.** *Suppose $\varphi(x,y)$ a formula, $A_n$,$B_n$ arbitrarily large matrices s.t.*

1. *There is a type which is satisfied by any tuple $(a_{ij}, b_{kl})$ for $i \neq k$ or $(i,j) = (k,l)$*
2. *The formula $\varphi(x,y)$ is satisfied by $(a_{ij}, b_{kl})$ if and only if $i \neq k$ or $(i,j) = (k,l)$*

*Then $T$ is non-equational. (Specifically, it is not $n$-equational for $|x| = n$)*

*Proof.* We will show that $\varphi$ is not a Boolean combination of equations. Assume the contrary. Then by Remark 2.3, $\varphi$ is equivalent to

$$\varphi(x,y) = \bigvee_{0\leq m\leq n} \left(\psi_1^m(x,y) \wedge \neg\psi_2^m(x,y)\right)$$

We have $\vDash \varphi(a_{11}, b_{11})$. So there is some $m_0$ such that if we define

$$\theta(x,y) := \left(\psi_1^{m_0}(x,y) \wedge \neg\psi_2^{m_0}(x,y)\right)$$

Then

$$\vDash \theta(a_{11}, b_{11}) = \left(\psi_1^{m_0}(a_{11}, b_{11}) \wedge \neg\psi_2^{m_0}(a_{11}, b_{11})\right)$$

Since $tp(a_{11}, b_{11}) = tp(a_{ij}, b_{kl})$ for $i \neq k$ or $(i,j) = (k,l)$ then we can deduce $\vDash \theta(a_{ij}, b_{kl})$ for such $i,j,k,l$.

If $i = k$ but $j \neq l$ then $\vDash \neg\varphi(a_{ij}, b_{kl})$ . In particular $\vDash \neg\theta(a_{ij}, b_{kl})$.

By Lemma 2.4 this means that $\theta$ is not a conjunction of an equation and a negation of an equation. Contradiction. □

# 3 Non-Equationality of $\mathbb{F}$

## 3.1 Working in $\mathbb{F}_\omega$

**Fact 3.1.** *(Sela) Non-abelian free groups share the same theory*

This fact allows us to work in $\mathbb{F}_\omega$. The motivation for it is so that we have a countable basis to create arbitrarily large matrices that satisfy the condition of Proposition 2.5 with the correct formula and type.

## 3.2 Definitions and useful properties of $\mathbb{F}$

**Definition 3.2.** (Reminder) An element of $\mathbb{F}$ is called *primitive* if it is part of some basis of $\mathbb{F}$.

**Fact 3.3.** *Let $a$ be a primitive element of $\mathbb{F}$. Suppose $a$ belongs to a subgroup $H$ of $\mathbb{F}$, then $a$ is a primitive element of $H$.*

*Proof.* Recall the Kurosh subgroup theorem:

**Fact 3.4.** *(Kurosh subgroup theorem) If $G = A * B$ and $H \leq G$, then:*

$$H = \left[\circledast_{A^g : g\in G} \left(H \cap A^g\right)\right] * \left[\circledast_{B^g : g\in G} \left(H \cap B^g\right)\right]$$

(**Note**: $\circledast$ is supposed to be a big asterisk $*$ representing the free product but I had some technical difficulties with it)

If $a$ is a primitive element and $a \in S$ then let $S$ be the rest of the basis to which it belongs. $\mathbb{F} = \langle a \rangle * \mathbb{F}(S)$ and so:

$$H = (H \cap \langle a \rangle) * \left[ \underset{\langle a \rangle^g : g \in \mathbb{F}}{\circledast} (H \cap \langle a \rangle^g) \right] * \left[ \underset{\mathbb{F}(S)^g : g \in \mathbb{F}}{\circledast} (H \cap \mathbb{F}(S)^g) \right]$$

Where $(H \cap \langle a \rangle) = \langle a \rangle \subseteq H$ and the rest of the free product is some free group with some basis $T$ so $\{a\} \cup T$ is a basis for $H$. □

**Fact 3.5.** *Let $e_1, \ldots, e_n$ be a basis of the free group $\mathbb{F}_n$ of rank $n$. Then $e_1^{m_1} \cdot e_2^{m_2} \cdots e_n^{m_n}$ is not a primitive element if for all $i$, $m_i \neq \pm 1$*

*Remark* 3.6. We had a similar argument for only one basis element $e_i^{m_i}$.

*Remark* 3.7. If $\exists i$ such that $m_i = \pm 1$ then $e_1^{m_1} \cdot e_2^{m_2} \cdots e_n^{m_n}$ is a primitive element. So in fact the condition in Fact 3.5 is *iff*.

## 3.3 Non-equationality of $\mathbb{F}_\omega$

We define the formula $\varphi_{ne}(x, y) = \forall u, v \left( [u, v] \neq 1 \to xy \neq u^5 v^4 \right)$

**Lemma 3.8.** *Let $\mathbb{F}_\omega := \langle e_1, e_2, \ldots \rangle$. Then for any pair $(a, b)$ which is part of some basis of $\mathbb{F}_\omega$ we have $\mathbb{F}_\omega \vDash \varphi_{ne}(a, b)$.*

*Proof.* It suffices to prove $(e_1, 1)$ satisfies $\varphi_{ne}$ since all primitive elements have the same type (there's an automorphism taking any basis to any other basis). So if $\vDash \varphi_{ne}(e_1, 1)$ then $\vDash \varphi(e_1 \cdot e_2, 1)$ (since $e_1 \cdot e_2$ is primitive) and so $\vDash \varphi(e_1, e_2)$ and so any two distinct basis elements satisfy $\varphi_{ne}$.

Assume $\exists u, v$, $[u, v] \neq 1$ such that $e_1 = u^5 v^4$. Then $\langle u, v \rangle$ is a free group of rank 2 (generated by two non-commuting elements) and so $e_1$ is a primitive element of $\langle u, v \rangle$ from Fact 3.3.

From Fact 3.5, $e_1$ is not a primitive element of $\mathbb{F}_\omega$. Contradiction. □

Define the two matrices for arbitrary $n \in \mathbb{N}$:

$$\mathbf{A}_n = \begin{pmatrix} e_2^5 e_1 & e_3^5 e_1 & \cdots & e_{n+1}^5 e_1 \\ e_3^5 e_2 & e_4^5 e_2 & \cdots & e_{n+2}^5 e_2 \\ \vdots & \vdots & \ddots & \vdots \\ e_{n+1}^5 e_n & e_{n+2}^5 e_n & \cdots & e_{2n}^5 e_n \end{pmatrix} \qquad \mathbf{B}_n = \begin{pmatrix} e_1^{-1} e_2^{-4} & e_1^{-1} e_3^{-4} & \cdots & e_1^{-1} e_{n+1}^{-4} \\ e_2^{-1} e_3^{-4} & e_2^{-1} e_4^{-4} & \cdots & e_2^{-1} e_{n+2}^{-4} \\ \vdots & \vdots & \ddots & \vdots \\ e_n^{-1} e_{n+1}^{-4} & e_n^{-1} e_{n+2}^{-4} & \cdots & e_n^{-1} e_{2n}^{-4} \end{pmatrix}$$

$$a_{ij} = e_{i+j}^5 e_i$$
$$b_{kl} = e_k^{-1} e_{k+l}^{-4}$$

**Lemma 3.9.** *Let $A_n = (a_{ij})$, $B_n = (b_{kl})$ as above. if $i \neq k$ or $(i, j) = (k, l)$ then $a_{ij}$ and $b_{kl}$ form part of a basis of of $\mathbb{F}_\omega$*

*Proof.* Consider first the case $i \neq k$. Extend $\{i, k\}$ by a subset $S \subseteq \{i + j, k + l\}$ of maximal size such that $S \cup \{i, k\}$ contains only pairwise distinct elements. Then the set $\{e_s | s \in S\} \cup \{a_{ij}, b_{kl}\}$ is part of a basis, as the subgroup it generates contains $\{e_i, e_k\} \cup \{e_s | s \in S\}$ which is a part of a basis of the same size.

If $(i, j) = (k, l)$ then the set $\{a_{ij}, b_{ij}\} = \left\{ e_{i+j}^5 e_i, e_i^{-1} e_{i+j}^{-4} \right\}$ forms a basis of $\mathbb{F}_2$ as the subgroup it generates contains $\{e_i, e_{i+j}\}$ which is part of a basis of the same size. □

**Lemma 3.10.** *Let $A_n = (a_{ij})$, $B_n = (b_{kl})$ as above. Then $\mathbb{F}_\omega \vDash \neg\varphi_{ne}$ by any pair $(a_{ij}, b_{kl})$ if $i = k$ and $j \neq l$.*

*Proof.* If $i = k$ take $a_{ij}$ and $b_{il}$ for $j \neq l$. Then:

$$a_{ij} b_{kl} = e_{i+j}^5 e_i e_i^{-1} e_{i+l}^{-4} = e_{i+j}^5 e_{i+l}^{-4}$$

$e_{i+j}, e_{i+l}^{-1}$ do not commute if $j \neq l$ so $\mathbb{F}_\omega \vDash \neg\varphi_{ne}(a_{ij}, b_{il})$. □

**Theorem 3.11.** *The theory of the free group is non-equational.*

*Proof.* By lemma 3.9 All pairs of the form $(a_{ij}, b_{kl})$ for $i \neq k$ and for $(i,j) = (k,l)$ are images of each other under automorphisms therefore they satisfy the same type. Namely $tp(e_1, e_2)$.

For the second condition, need to show that $\mathbb{F}_\omega \vDash \varphi_{ne}(a_{ij}, b_{kl})$ iff $i \neq k$ or $(i,j) = (k,l)$.

From lemma 3.9 if $i \neq k$ or $(i,j) = (k,l)$ then $a_{ij}, b_{kl}$ are a part of a basis and so from lemma 3.8 $\mathbb{F}_\omega \vDash \varphi_{ne}(a_{ij}, b_{kl})$. In the other direction, if $\mathbb{F}_\omega \vDash \varphi_{ne}(a_{ij}, b_{kl})$ then from lemma 3.10 we have $i \neq k$ or $(i,j) = (k,l)$.

So the conditions of Proposition 2.5 hold and the theory of $\mathbb{F}_\omega$ is non-equational. □

# 4 Non-equationality of free product of groups

## 4.1 Motivation

**Fact 4.1.** *(Sela) Let $G_1 * G_2$ be a nontrivial free product which is not $\mathbb{Z}_2 * \mathbb{Z}_2$. Then it is elementarily equivalent to $G_1 * G_2 * \mathbb{F}$ for* any *free group $\mathbb{F}$.*

**Fact 4.2.** *(Sela) A free product of stable groups is stable.*

From these facts it suffices to show that the theory of a free product $G * \mathbb{F}_\omega$ is not equational to get a zoo of other examples of non-equational stable theories.

**Theorem 4.3.** *Let $G_1 * G_2$ be a nontrivial free product which is not $\mathbb{Z}_2 * \mathbb{Z}_2$. Then its first order theory is non-equational.*

To prove this we will prove the following lemma:

**Lemma 4.4.** *Let $\mathbb{F}_\omega = \langle e_1, e_2, \ldots, e_n, \ldots \rangle$. Then for any pair $(a,b)$ which is part of some basis of $\mathbb{F}_\omega$ we have that $G * \mathbb{F}_\omega \vDash \varphi_{ne}(a,b)$.*

If we prove this lemma then, remembering that basis elements of $\mathbb{F}_\omega$ inside $G * \mathbb{F}_\omega$ still have the same type, and that lemma 3.10 still holds, it would mean by Proposition 2.5 that theorem 4.3 is true.

## 4.2 Bass-Serre Theory

**Definition 4.5.** If $G = G_1 * G_2$ We call an expression of the form $g := g_1 g_2 \cdots g_n \in G$ a *normal form* if $g_i \in (G_1 \cup G_2) \setminus \{1\}$ and no two consecutive components are in the same $G_i$.

**Fact 4.6.** *This form is unique.*

**Definition 4.7.** For $g \in G$ with normal form $g_1 g_2 \cdots g_n$, the *syllable length* of $g$ is defined $\mathrm{syl}(g) := n$.

**Fact 4.8.** *The identity element is the unique element with syllable length 0*

**Definition 4.9.** An element $g$ with normal form $g_1 g_2 \cdots g_n$ is called cyclically reduced if $g_1, g_n$ lie in different groups $G_i$.

**Fact 4.10.** *Any element $g \in G$ can be written as $\gamma g' \gamma^{-1}$ where $g'$ is cyclically reduced.*

Recall Bass-Serre Theory.

Elements of $G = G_1 * G_2$ act on a tree where vertices are cosets $gG_i$ for $i = 1,2$ and an edge exists between $gG_1$ and $gG_2$.

Any edge is trivially stabilized.

$\mathrm{Stab}(gG_i) = G_i^g$.

Elements are either elliptic or hyperbolic.

Nontrivial elliptic elements $h$ stabilize a unique vertex $\mathrm{Fix}(h) = gG_i$.

Hyperbolic elements $h$ have an infinite line $\mathrm{Ax}(h)$ (the axis of $h$) on which $h$ acts by translation by some fixed length $\mathrm{tr}(h) > 0$.

*Remark* 4.11. Let $u, v$ be hyperbolic elements in $G$ such that their axes intersect in length at least $\mathrm{tr}(u) + \mathrm{tr}(v) + 1$. Then $u, v$ commute.

**Fact 4.12.** *If $g, g' \in G_1 * G_2$ are both elliptic and* $\mathrm{Fix}(g') \neq \mathrm{Fix}(g)$ *then $gg'$ is hyperbolic with* $\mathrm{tr}(gg') = 2d(\mathrm{Fix}(g), \mathrm{Fix}(g'))$

*Remark* 4.13. We can think of an elliptic element $g \in G_1 * G_2$ as having $\mathrm{tr}(g) = 0$ and its axis $\mathrm{Ax}(g)$ consists of the point $\mathrm{Fix}(g)$.

**Fact 4.14.** *Let $u \in G_1 * G_2$ be a cyclically reduced hyperbolic element. Then its axis* $\mathrm{Ax}(u)$ *contains $G_1$ and $G_2$.*

*Remark* 4.15. If $u \in G_1 * G_2$ is a hyperbolic element which is not cyclically reduced, then $u = \gamma u' \gamma^{-1}$ and its axis is a translation of $\mathrm{Ax}(u')$ by $\gamma$.

## 4.3 Proof of Lemma 4.4

As in the proof of lemma 3.8 we need to show that for non-commuting $u, v \in G * \mathbb{F}_\omega$ then $u^5 v^4 \neq e_1$.

### 4.3.1 Reduction to a special case

First we take note that it suffices to assume in the lemma that one of $u, v$ is cyclically reduced:

**Lemma 4.16.** *Assume the criterion for lemma 4.4 holds for any elements $u, v$ which do not commute and where at least one of $u, v$ is cyclically reduced, then it holds for all $u, v$.*

*Proof.* If $u^5 v^4 = e_1$ then in the normal form either $u$ starts with the *letter* $e_1$ or $v$ ends with $e_1$. Assume $u$ starts with $e_1$ (proof symmetric otherwise) and neither $u, v$ are cyclically reduced. Then $u, v$ are conjugates of cyclically reduced non-trivial words and more precisely $u = e_1 u' e_1^{-1}$ and $v = v_1 v' v_1^{-1}$ with $v_i$ some element in $G$ or a letter in $\mathbb{F}_\omega$ ($u', v'$ not necessarily cyclically reduced) whence

$$u^5 v^4 = e_1 u'^5 e_1^{-1} v_1 v'^4 v_1^{-1}$$

If $v_1 \neq e_1$ then there is no cancellation in the product so $u^5 v^4$ cannot equal $e_1$. Hence $e_1 = v_1$ and so:

$$u^5 v^4 = e_1 u'^5 v'^4 v_1^{-1} = e_1 \iff u'^5 v'^4 = e_1$$

And $u'^5 v'^4$ still commute. We can decrease $u', v'$ further in this manner, and by a length argument we will reach some commuting $v_0, u_0$ where $v_0^5 u_0^4 = e_1$ and at least one of them cyclically reduced, contradicting the assumptions. □

### 4.3.2 Proof of Lemma 4.4

*Proof.* As we mentioned, It suffices to show that $e_1$ cannot be written as $u^5 v^4$ for some elements $u, v$ that do not commute, and where one of $u, v$ is cyclically reduced. Assume such $u, v$ existed. Consider their action on the Bass-Serre tree corresponding to the splitting $G * \mathbb{F}_\omega$

We prove for the case that $v$ is cyclically reduced and starts with a syllable from $G$. The case where $u$ is cyclically reduced and starts with a syllable in $\mathbb{F}_\omega$ is the same. The other cases are symmetrical.

*Case* 1. Assume both $u$ and $v$ are elliptic. Then if $\mathrm{Fix}(u) \neq \mathrm{Fix}(v)$ then $u^5 v^4$ is hyperbolic by fact 4.12 and so cannot be $e_1$ which fixes $\mathbb{F}_\omega$.
If $\mathrm{Fix}(u) = \mathrm{Fix}(v)$ then this vertex is $\mathrm{Fix}(e_1) = \mathbb{F}_\omega$ and hence $u, v \in \mathbb{F}_\omega$, non-commuting so they generate a free group of rank 2. From fact 3.5 this means that $e_1$ cannot be primitive in this group and so from fact 3.3 it is not primitive in $\mathbb{F}_\omega$. Contradiction.

*Case* 2. Assume $v$ is elliptic. Then $u$ is hyperbolic. $v$ must fix either $\mathbb{F}_\omega$ or $G$. In the former case $u^5 = e_1 v^{-4}$ fixes $\mathbb{F}_\omega$, contradicting $u$ being hyperbolic, and in the latter, by lemma 4.12 $u^5 = e_1 v^{-4}$ has $\mathrm{tr}(u^5) = 2$ while we know $\mathrm{tr}(u^5) = 5\mathrm{tr}(u) \geq 5$. Contradiction.

*Case* 3. Assume $v$ is hyperbolic. We can think of $u$ being elliptic as a special case of $u$ being hyperbolic with $\mathrm{tr}(u) = 0$. so the rest of the proof will assume both $v, u$ are hyperbolic.

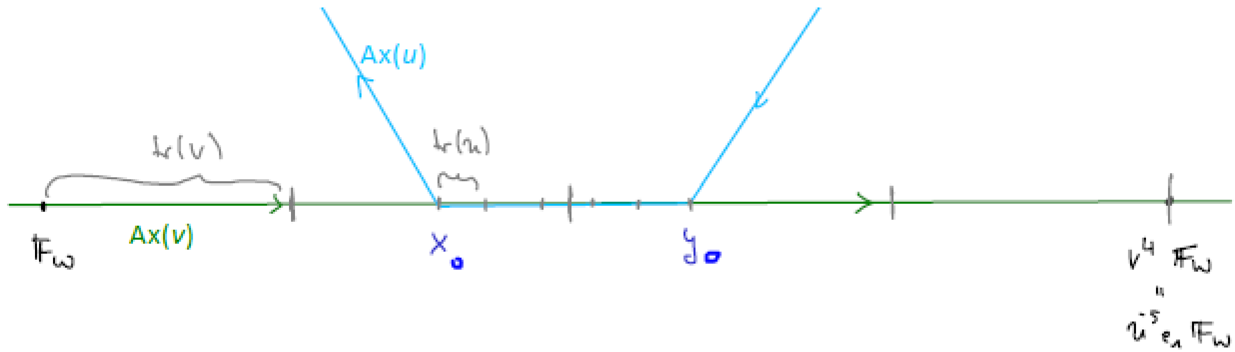


Figure 1: Axes of $v, u$ and the action on $\mathbb{F}_\omega$ by $v^4 = u^{-5}e_1$ where $v$ is hyperbolic and cyclically reduced

Let $v = b_1 a_1 b_2 \cdots b_n a_n$ in the normal form, where $b_i \in G$ and $a_i \in \mathbb{F}_\omega$. Let $u_1 u_2 \cdots u_m$ be the normal form of $u^5$. If $u$ is also cyclically reduced then $\mathrm{Ax}\,(u)$ and $\mathrm{Ax}\,(v)$ coincide for strictly more than $\mathrm{tr}\,(u) + \mathrm{tr}\,(v)$ since the path from $\mathbb{F}_\omega$ to $v^4\mathbb{F}_\omega = u^{-5}\mathbb{F}_\omega$ lies in both axes and it is of length at least $4\max\{\mathrm{tr}\,(u), \mathrm{tr}\,(v)\} > \mathrm{tr}\,(u) + \mathrm{tr}\,(v)$, whence by remark 4.11 they commute. Contradiction.

Assume that $u$ is not cyclically reduced then. In particular $u_1, u_m \in G$ or $u_1, u_m \in \mathbb{F}_\omega$. The latter cannot hold since then there is no cancellation in $u^5 \cdot v^4$ and hence it cannot be $e_1$.

Now in the Bass-Serre tree $\mathbb{F}_\omega$ is moved by $v^4$ along its axis to $y = v^4\mathbb{F}_\omega$ that is labeled:

$$y = v^3 b_1 a_1 b_2 \cdots b_n a_n \mathbb{F}_\omega = v^3 b_1 a_1 b_2 \cdots b_n \mathbb{F}_\omega$$

We assume the axes of $u$ and $v$ coincide for at most $\mathrm{tr}\,(u) + \mathrm{tr}\,(v)$. Otherwise, as above, they commute and we're done. This implies that each of the two parts of the axis of $v$ outside the axis of $u$ between $x$ and $y$ is of length at least $\mathrm{tr}\,(v) = \mathrm{syl}\,(v) = 2n$: The action of $u^{-5}$ on $x$ is as follows: It takes $x$ to some point $x_0$ on $\mathrm{Ax}\,(u) = \mathrm{Ax}\,(u^5)$, translates it by $5\mathrm{tr}\,(u)$ to some point $y_0$ and then sends it to $y$ by the same length as $d\,(x, x_0)$. So the axes of $u, v$ coincide for a total length of $c = 5\mathrm{tr}\,(u) \le \mathrm{tr}\,(u) + \mathrm{tr}\,(v)$ and:

$$d\,(x, x_0) = d\,(y_0, y) = \frac{d\,(x, y) - d\,(x_0, y_0)}{2} = \frac{4\mathrm{tr}\,(v) - c}{2} \ge \frac{3\mathrm{tr}\,(v) - \mathrm{tr}\,(u)}{2} \ge \mathrm{tr}\,(v) = 2n$$

Where $\mathrm{tr}\,(v) \ge \mathrm{tr}\,(u)$ since $4\mathrm{tr}\,(v) \ge 5\mathrm{tr}\,(u) \ge 4\mathrm{tr}\,(u)$. (See figure 1)

Now, since $u^5 v^4 = e_1$,we must have that $u^{-5}$ moves $x$ to $y$ along the axis of $v$. Hence $y$ is also labeled

$$y = u^{-5}\mathbb{F}_\omega = u_m^{-1} \cdots u_2^{-1} u_1^{-1}$$

and since $u_1^{-1} \in G$ we deduce $u_1^{-1} = b_n$. We can repeat this argument $2n$ times, getting:

$$u_i^{-1} = \begin{cases} b_{n-\frac{i-1}{2}} & (i \text{ odd}) \\ a_{n-\frac{i}{2}} & (i \text{ even}) \end{cases}$$

Where $a_0 := a_n$ (The last syllable of the previous instance of $v$)

Since $u$ is not cyclically reduced, it is of the form $\gamma u' \gamma^{-1}$ where $u'$ is cyclically reduced and $\mathrm{syl}\,(\gamma)$ is at least $2n$. Thus $u_{m-i+1} = u_i^{-1}$ for $i \le 2n$. Now, on the axis of $v$, walking $2n$ steps s starting at $x$ we have:

$$\begin{array}{cccccc} \mathbb{F}_\omega, & G, & u_m^{-1}\mathbb{F}_\omega, & u_m^{-1}u_{m-1}^{-1}G, & \ldots & u_m^{-1}u_{m-1}^{-1}\cdots u_{m-2n+1}^{-1}G \\ \mathbb{F}_\omega, & G, & u_m^{-1}\mathbb{F}_\omega, & b_1 a_1 G, & \ldots & b_1 a_1 \cdots b_n a_n G \end{array}$$

All in normal forms. Then the uniqueness in normal forms implies $a_n = u_{m-2n+1}^{-1} = u_{2n} = a_n^{-1}$. A contradiction, since $a_n^{-1}$ is a nontrivial element in $\mathbb{F}_\omega$. This concludes that $u^5 v^4 \neq e_1$ thus proving this case.

□