# LECTURE 7: THE WEIRDO AMONG STABLE GROUPS

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Before Sela added the free group to the class of stable groups, abelian groups and algebraic groups were the two leading examples of the class and it was the intuition that much of general truth about stable groups could be deduced from studying these two classes alone. Selas result heavily changed that picture of stable groups, not only because the free group was not quite an expected candidate for being stable, but also because there are several notions which hold for abelian and algebraic stable groups, and were thus expected to hold for stable groups as such, and which were contradicted by the free group. We will see and discuss three of these notions in the talk.

## 1. The Tricotomy Conjecture

We noted that *morally*, abelian and algebraic groups were the only pure stable groups considered. While in the stable setting, this was *morally* true, in some stronger class it is still not known if there are other examples.

**Conjecture 1.1** (Algebraicity Conjecture)**.** *Every infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field.*

Recently, it has been shown that the conjecture is true for groups of Morley rank up to 3. If you manage to solve it, fame will be yours.
The Algebraicity Conjecture is intimately related to another famous conjecture, *Zil'bers Trichotomy Conjecture*. We will repeat some notions from the exercise class.

**Definition 1.2.** We say that a set $X$ together with some function $\mathrm{cl}: \mathcal{P}(X) \to \mathcal{P}(X)$ is a *pregeometry*, if the following conditions hold:

- $A \subseteq \mathrm{cl}(A), A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B), \mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$
- $b \in \mathrm{cl}(A)$ if and only if there exists some finite $A_0 \subseteq A$ such that $b \in \mathrm{cl}(A_0)$ and
- **Exchange:** If $a, c$ are elements such that $a \in \mathrm{cl}(Bc) \setminus \mathrm{cl}(B)$ for some set $B$, then also $c \in \mathrm{cl}(Ba)$.

If $(X, \mathrm{cl})$ is a pregeometry, we define for any finite subset $A \subseteq X$ a **dimension**

$$d(A) := \min\{|A_0| \mid A_0 \subseteq A \subseteq \mathrm{cl}(A_0)\}.$$

**Fact 1.3.** *If a theory $T$ is strongly minimal, i.e. in any of its models every definable set is either finite or co-finite, then the algebraic closure operator defines a pregeometry on each of the models of $T$.*

**Remark 1.4.** Note that the dimension function $d$ is **submodular**, i.e. for all $A$ and $B$ finite dimensional and closed, we have

$$d(AB) \leq d(A) + d(B) - d(A \cap B).$$

If the inequality is an equality (whenever $d(A \cap B) > 0$), then we call $d$ **(locally) modular**.
On the other hand, for any such formula we can define a closure operator $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ via

$$\mathrm{cl}(A) = \{x \in X \mid \exists A_0 \underset{\text{fin}}{\subseteq} A \text{ such that } d(A_0) = d(A_0 x)\}.$$

**Examples 1.5.**

- The pure set, $\mathrm{cl}(A) = A$. What is the dimension?
- infinite dimensional vector spaces. What is the dimension?
- Algebraically closed fields. What is the dimension?

The famous Trichotomy Conjecture now suggested that these three examples would essentially cover all strongly minimal pregeometries.

**Conjecture 1.6** (Zilbers Trichotomy Conjecture)**.** *The acl-geometry of some strongly minimal set $M$ falls into one of the following three disjoint classes:*

*(i) It is set like, i.e. disintegrated, i.e.* $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$.
*(ii) It is vector space like, i.e. not disintegrated, but locally modular.*
*(iii) There is an algebraically closed field interpretable in the theory of $M$.*

Note the strong implication Zil'bers Conjecture would have in the realm of interaction between model theory and algebra: If we start with an arbitrary strongly minimal theory whose geometry is not locally modular, then an actual field is interpretable in it and hence we can do algebraic geometry. Nevertheless, the conjecture had been refuted by Hrushovski [**?**] soon after. Nevertheless, work around the Trichotomy Conjecture continued and proved to be a rich source of research. The continuation of Zil'bers Conjecture can basically be partitioned into two streams:

(1) What further conditions should be put in order to make the conjecture hold?
(2) How far is the conjecture from being true? Can we fill the gap between vectorspace-like geometries and algebraically closed fields?

It turns out that there is a very natural setting in which Zil'bers Conjecture holds: the Zariski geometries. The establishment of Zil'bers Conjecture in this setting had far reaching consequences and allowed Hrushovski to give model theoretic proofs of profound number theoretic conjectures.

**Example 1.7** (DCF and Mordel-Lang)**.** Roughly speaken, the Mordel-Lang conjecture states that given an abelian variety, the intersection of a proper subvariety with a subgroup is a finite union of subgroups. In order to tackle this conjecture in a model theoretic approach, Hrushovski considered the underlying fields in the language of rings enlarged by a symbol $\partial$ for a difference function, which is additive and satisfies $\partial(xy) = x\partial(y) + y\partial(x)$. This leads to the theory of *differential fields*. Its model companion is the theory of differentially closed fields *DCF*. Now Hrushovski proved that in a differentially closed field any type of rank one is either one-based, which is the analog of locally modular for arbitrary simple theories, or it is non-orthogonal to the (algebraically closed) field of constants, i.e. the field of all elements such that $\partial(x) = 0$. Thus he established a version of Zilbers Conjecture in the framework of differentially closed fields. This lead him to give a proof of the famous Mordel-Lang Conjecture in arbitrary characteristic.

**Example 1.8** (ACFA and Munin-Mumford)**.** The Munin-Mumford conjecture, again resting deep inside the area of number theory, can be seen as an analog of the Mordel-Lang conjecture, where the subgroup is the group of torsion elements of the abelian variety. To consider this conjecture from a model theoretic point of view, one considers the underlying field in the language of ring enriched by one symbol $\sigma$ for an automorphism. This leads to the theory of *difference fields*. Its model companion is *ACFA*, the theory of algebraically closed fields with a generic automorphism. Here, Chatzidakis and Hrushovski again established Zil'bers Conjecture, proving that any type of rank 1 is either one-based or almost internal to some field, which is either the field $\operatorname{Fix}(\sigma)$ or the field $\operatorname{Fix}(\sigma^n \operatorname{Frob}_p^n)$, for positive characteristic, where $\operatorname{Frob}_p$ is the Frobenius automorphism, sending an element to its $p$-th power. This lead them to give a completely model theoretic proof of the Munin-Mumford Conjecture.

## 1.1. **Ampleness.**

**Definition 1.9.** Let T be a stable theory and $n \in \mathbb{N}$ arbitrary. We say that a theory is $n$-ample if possibly after naming parameters there are tuples $a_0, a_1 \dots, a_n$ which satisfy the following properties:

(i) We have $\operatorname{acl}^{eq}(a_0) \cap \operatorname{acl}^{eq}(a_1) = \operatorname{acl}^{eq}(\emptyset)$;
(ii) For all $1 \leq i < n$ it holds $\operatorname{acl}^{eq}(a_0, \dots, a_{i-1} a_i) \cap \operatorname{acl}^{eq}(a_0, \dots, a_{i-1}, a_{i+1}) = \operatorname{acl}^{eq}(a_0, \dots, a_i)$;
(iii) For all $1 \leq i < n$ we have $a_0, \dots, a_{i-1} \mathop{\smile\!\!\!\!\!\mid}_{a_i} a_{i+1}$ and
(iv) It holds that $a_0 \not\mathop{\smile\!\!\!\!\!\mid} a_n$.

One can understand the degree of ampleness of being a measure on how complicated the forking relation is within the given theory. We already observed that in vector spaces the independence is completely described by the algebraic closures. One can understand the ampleness of a theory in a way of a measure, how far this statement is from being true.
The following remark is easy to see.

**Remark 1.10.** (1) The notions of ampleness form a hierarchy, i.e. any structure which is $n+1$ ample, is also $n$ ample.
(2) A stable theory T is 1-ample, if and only if it is not one-based.
(3) A stable theory T is 2-ample, if and only if it is not CM-trivial.

Current research suggests, we should not consider this notion for all stable theories, but rather for all pregeometries. The algebraic closure as closure operator is too week in strictly stable theories and the notion might gain essence by moving to another notion of closure which actually satisfies exchange.

**Fact 1.11.** *Vector spaces and pure Abelian Groups are one-based, i.e. not* 1*-ample. Algebraically closed fields are* $n$*-ample for all natural numbers* $n$*.*

The notion of ampleness was introduced by Pillay. He also proved that the free group is 2-ample and conjectured that this would be its maximal level of ampleness. Alas...

**Fact 1.12** (Sklinos)**.** *The free group is $n$-ample for any natural number $n$.*

This property was considered to be inert to algebraic groups and algebraically closed fields. Nevertheless,

**Fact 1.13** (Byron, Sklinos)**.** *There is no infinite field definable in $\mathbb{F}$.*

This is the only known example of a pure group which is $n$-ample for all $n$ and does not interpret a field.

## 2. Equationality

Rumor has it, that Sela asked Hrushovski which kind of results about the free group would interest model theorists. When the latter mentioned stability and Sela asked how one could prove something like this, Hrushovski replied: "You could try to prove that it is equational." Until then, there had only been one non-equational, stable structure, which was constructed by Hrushovski himself in a rather complicated combinatorial matter. It seemed that in all reasonable cases, equationality and stability would fall together. Nevertheless, some few days later, Sela came back to Hrushovski, saying: "Well, it is not equational. How else can you show stability?"
As so many notions in Model Theory, the notion of equationality takes its inspiration from core math and, once again, specifically from Algebraic Geometry. It can be understood as a version of Noetherianity for the first order setting.

**Definition 2.1** (Noetherian)**.** A topological space is called *noetherian*, if the family of closed sets has the descending chain condition, i.e. any descending sequence of closed sets stabilizes eventually.

Zarriski topologies are noetherian due to the fact that there is a dimension on its set and proper subspaces drop in that dimension.
Now we want to give the model theoretic analogue of noetherianity. As frequent in model theory, the notion of an equation is introduced for formulas and transferred from there to arbitrary theories.

**Definition 2.2.** Work in a theory $T$ and saturated models $M$ of it. We say that a formula $\varphi(x, y)$ is an **equation** (in x), if the family of its instances has the descending chain condition. This means that for any $I$ and $(a_i)_{i\in I} \in M$, there is some $I_0 \subseteq I$ such that $\bigcap_{i\in I} \varphi(M, a_i) = \bigcap_{i\in I_0} \varphi(M, a_i)$.
We say that the theory $T$ is $n$-equational, if any of the formulas $\varphi(x, y)$ where $x$ is an $n$-tuple, is a Boolean combination of equations. We say that $T$ is equational, if it is $n$-equational for all $n \in N$.

**Remark 2.3.** Zarriski Tpology: Note that in algebraically varieties, the definable sets are Boolean combinations of Zarriski closed sets, which are instances of equations by Noetherianity. Hence, algebraically closed fields are equational exactly because there are Noetherian.

**Exercise 2.4.** Equational implies stable.

What about the converse ? Stability yields a notion of independence. Can we hope for an analogue of a "drop of dimension" as in Zarriski geometries which ensures equationality? Let us confer with the expert on counter examples...

**Example 2.5** (Hrushovski - Srour)**.** There is an $\omega$-stable structure of infinite Morley rank which is not equational.

So again, what if groups enter the picture? Our two original prototypical examples of stable groups, do not help much.

**Fact 2.6.** *(i) ACFs are equational.*
*(ii) Abelian Groups are equational. Indeed, any one based theory is equational.*

So one could conjecture that any stable group is also equational. Once again, the free group troubles this picture.

**Fact 2.7.** *The first order theory of the free group is not equational.*

The proof is very involved and uses tower constructions and test sequences. It has not yet been published. There is an easy criterion which yields a stronger result (building on a very deep theorem of Sela):

**Fact 2.8.** *Any theory of the free product of stable groups, not both isomorphic to $\mathbb{Z}_2$, is stable (Sela) and non-equational.*

**Remark 2.9.** Note that there are many easily accessible open questions about equationality:

- Are strongly minimal sets equational?
- Does 1-equationality imply equationality?
- How does the ample hierarchy relate to equationality?
- Under which conditions do equationality and nfcp fall together?

In case you want to start your own research.

## 3. The Finite Cover Property

Another sharpening of stability is given by a theory Not having the Finite Cover Property (NFCP). In the stable case, this is equivalent to eliminating the $\exists^\infty$ quantifier. The notion of NFCP is considered to go hand in hand with the notion of equationality and the free group was thus expected to indeed have the finite cover property. As we reach the end of this lecture, we should not be surprised by the following theorem:

**Fact 3.1.** *The free group does not have the finite cover property.*

Anyway, NFCP gives that probably free group with generic automorphism exists, and theory of belle pairs (starated models are again belle pairs).

**Definition 3.2.** We say that a formula $\varphi(x, y)$ has the finite cover property (fcp), if there does NOT exist any $k \in \mathbb{N}$ such that for any sequence $(a_i)_{i\in I}$ we have that $\{\varphi(x, a_i \mid i \in I\}$ is $k$-consistent if and only if it is consistent. We say that a theory does not have the finite cover property, if non of its formulas has the finite cover property.

**Fact 3.3.** *Any theory which is NFCP, is also stable. On the other hand, there are $\omega$-stable theories, which do have FCP.*

Another characterisation of having NFCP, when we already know that the ambient theory is stable, is given by the following:

**Fact 3.4.** *A stable theory $T$ has NFCP if and only if it eliminates the $\exists^\infty$ quantifier.*

Recall that we call a theory $T$ **categorical** in some cardinal $\kappa$, if it has (up to isomorphism) exactly one model of cardinality $\kappa$.

**Fact 3.5.** *If $T$ is categorical in some cardinality $\kappa$, then it has nfcp.*

Remember the beginning of the lecture where we mentioned in Morleys categoricity theorem that algebraically closed fields are uncountably categorical. This yields the following Corollary.

**Corollary 3.6.** *Algebraically closed fields don't have the finite cover property.*

**Exercise 3.7.** Are algebraically closed fields also $\aleph_0$ categorical?

Also abelian groups have NFCP, though following another argument. The free group provides the only known example of a stable structure, which does not have the finite cover property, but nevertheless is not equational, proving that NFCP does not imply equationality.
For those acquainted with the notions, we want to point out that there are strong consequences of a theory having NFCP:

**Remark 3.8.** If T has NFCP, then the theory of its belle paires $T_P$ and the theory of adding a generic automorphism $TA$ exist.