Lecture 4 - JSJs and Makanin Razborov diagram

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Let $\mathbb{F} = \mathbb{F}_n$ denote the free group on $a_1, \dots, a_n$.

1 Equations in the free group

Simplest example of first order formula in the language of groups: equation. Ex: eqn in two variables x, y is of the form $w(x, y) = 1$ where w is a (reduced) word on x, y (and $a_1, \dots, a_n$ if we have parameters). The corresponding definable set is the subset of $G \times G$ consisting of all pairs (g, h) such that $w(g, h) = 1$.

Example 1.1: $xyx^{-1}y^{-1} = 1$
$x^3 = a_1 a_2$

Suppose $\Sigma(x_1, \dots, x_k) = 1$ is a system of equations in k variables.

Definition 1.2: *Denote V_Σ the set of solutions to Σ in $\mathbb{F}$. The group G_Σ given by the presentation $\langle x_1, \dots, x_k \mid \Sigma(x_1, \dots, x_k) \rangle$ is called the coordinate group of V_Σ.*

Then there is a one-to-one correspondence between points in V_Σ and homomorphisms $G_\Sigma \to \mathbb{F}$.

The following proposition will help us analyze the set of morphisms $G \to \mathbb{F}$, and thus to understand sets defined by equations:

Proposition 1.3: *Let G be a finitely generated group, which is freely indecomposable.*

Then there is a finite set of proper quotients $\eta_1 : G \to L_1, \dots, \eta_r : G \to L_r$ such that any homomorphism $h : G \to \mathbb{F}$ factors through one of the η_i's ***after precomposition by an element of*** $\mathrm{Aut}(G)$*, i.e for some i there exists $h' : L_i \to \mathbb{F}$ and $\sigma \in \mathrm{Aut}(G)$ such that $h \circ \sigma = h' \circ \eta_i$.*

We also have a relative version of the same result

Proposition 1.4: *Let G be a finitely generated group. Let $A \leq G$ be such that G is freely indecomposable with respect to A. Let $h_A : A \to \mathbb{F}$ be a fixed morphism.*

Then there is a finite set of proper quotients $\eta_1 : G \to L_1, \dots, \eta_r : G \to L_r$ such that any homomorphism $h : G \to \mathbb{F}$ ***whose restriction to A is exactly*** *h_A factors through one of the η_i's* ***after precomposition by an element of*** $\mathrm{Aut}_A(G)$*, i.e for some i there exists $h' : L_i \to \mathbb{F}$ and $\sigma \in \mathrm{Aut}(G)$ such that $h \circ \sigma = h' \circ \eta_i$.*

This enables us to prove:

Theorem 1.5: *(Sela) Let G be a finitely generated group. There is a finite tree of groups rooted at G, such that*

1. *each pair L, L' where L' is a child of L comes with a quotient map $\eta : L \to L'$, which is proper except possibly if $L = G$;*
2. *the leaves are free group;*
3. *for any morphism $h : G \to \mathbb{F}$, there is a branch $G \to L_1 \to L_2 \to \dots \to L_s \to F$ and automorphisms σ_i of the L_i, and an injective map $j : F \hookrightarrow \mathbb{F}$ such that*

$$h = j \circ \sigma_s \circ \eta_{s-1} \circ \dots \circ \eta_1 \circ \eta_0$$

Idea of the proof: for each L appearing, decompose L into free factors $L = L^1 * L^2 * F$ where L^1, L^2 not free and F is free. Apply Proposition above to L^1, L^2 to get quotients $\eta_1 : L^1 \to R_1, \ldots, eta_r : L^1 \to R_r$ and $\mu_1 : L^2 \to Q_1, \ldots, \mu_q : L62 \to Q_q$ - then define the children of L to be the quotients $R_i * Q_j * F$ of L, with quotient maps $\eta_i * \mu_j * \mathrm{Id}_F$.

Definition 1.6: *This tree is called the Makanin-Razborov diagram.*

But in fact we know a little bit more about the groups appearing (they are all limit groups) and the automorphisms σ_i used - if $L_i = L_i^1 * \ldots * L_i^t$ then $\sigma_i \mid_{L_i^k}$ is a **modular** automorphism.

2 The modular group

Definition 2.1: *Amalgamated products, HNN extensions. Corresponding Dehn twists. Relative version.*

We can generalize the notion of amalgamated products and HNN extension to that of a graph of groups.

Definition 2.2: *A **graph of group** $\mathcal{G}$ consists of a graph Γ to gether with the following data:*

1. *for each vertex v, a vertex group G_v;*
2. *for each edge e, an edge group G_e and embeddings $i_0 : G_e \to G_{o(e)}$ and $i_1 : G_e \to G_{t(e)}$ (so that $G_{\bar{e}} \simeq G_e$ and the embeddings are swapped).*

*The **fundamental group** $\pi_1(\mathcal{G})$ of a graph of group $\mathcal{G}$ is the group generated by the vertex groups together with a set $\{t_e \mid e \in E(\Gamma)\}$ and the relations:*

- *$t_{\bar{e}} = t_e^{-1}$ for all edges e*
- *$i_0(g) = t_e i_1(g) t_e^{-1}$ for all edges e;*
- *$t_e = 1$ for all edges $e \in E(\Gamma_0)$;*

where Γ_0 is a maximal subtree of Γ. It is possible to show that the isomorphism class of $\pi_1(\mathcal{G})$ does not depend on the choice of Γ_0, and that the groups G_v and G_e embed in $\pi_1(\mathcal{G})$ (Bass-Serre theory).

Note that if $G = \pi_1(\Gamma)$, can recover from each edge of Γ a splitting of G as an HNN or an amalgamated product.

Question: is there a graph of group from which one can recover all the splittings of G in this way? Or say, all splittings for which edge groups are cyclic groups? abelian groups?

Remark 2.3: *Splittings are not always compatible! (surface group example)*

Under some circumstances however, this is the only obstruction, i.e. if there are some splittings which are not compatible, there is a surface somewhere.

We will show this is the case when G is a torsion free hyperbolic group (for example the free group) which is freely indecomposable (relative to a subgroup A).

3 JSJ decompositions and modular automorphisms in the case of a torsion free hyperbolic group

Definition 3.1: *Let $\mathcal{G}$ be a graph of groups. We say that v is a **surface type vertex** in $\mathcal{G}$ if G_v is the fundamental group of a surface with boundary of characteristic at least -2 or a punctured torus, and the map $e \to [i_1^e(G_e)]$ induces a bijection between the set of edges adjacent to v and the set of conjugacy classes of boundary subgroups in G_v.*

In other words, G_v admits a presentation of the form

$$\langle b_1, \ldots, b_r, x_1, y_1, \ldots, x_g, y_g, (z) \mid b_1 \ldots b_r [x_1, y_1] \ldots [x_g, y_g](z^2) = 1\rangle$$

and the edges adjacent to v are $e_1, \dots e_r$ with $G_{e_j} = \langle \beta_j \rangle$ and $i_1^{e_j}(\beta_j) = b_j$.

We have:

Theorem 3.2: *Let G be a torsion free hyperbolic group (for example a free group) which is freely indecomposable (with respect to a subgroup A). There exists a graph of groups decomposition $\mathcal{G}$ for G with infinite cyclic edge groups and a set V_S of vertices of surface type such that any splitting of G as amalgamated product or an HNN extension over an infinite cylic group can be obtained from Γ by*

1. *possibly refining a surface type vertex group by a splitting induced by a simple closed curve;*
2. *collapsing all the edges but one.*

In that case we can also give an easy description of the "modular group" that appears in the precise version of Theorem 1.5.

Definition 3.3: *In the case of a torsion free hyperbolic group G which is freely indecomposable (relative to a subgroup A), the modular group is defined to be the subgroup of $\mathrm{Aut}_A(G)$ generated by all the Dehn twists corresponding to all possible splittings of G as an HNN or an amalgamated product over an infinite cyclic group.*

Because the JSJ encodes all such possible splittings (in this case) then we have the following:

Corollary 3.4: *Modular automorphisms restrict to conjugation on non surface and non abelian type vertex groups of the JSJ decomposition, and send surface type vertex groups isomorphically on conjugates of themselves.*

Another key result obtained by Rips and Sela

Theorem 3.5: *If G is a torsion free hyperbolic group which is freely indecomposable (relative to a subgroup A), the modular group $\mathrm{Mod}_A(G)$ has finite index in the full group of automorphisms $\mathrm{Aut}_A(G)$.*