LECTURE 2 INTRODUCTION TO STABLE GROUPS

ISABEL

1. Some Model Theory

Recall (Sela): All non-abelian free groups have the same first order theory, which is stable.

Definition 1.1 (Stability)**.** no order property

Remark 1.2. notion by Shelah, classification of first oder theories into rather tame ones (we have independence, forking, algebraic closure, "few types") and wild ones.

1.1. In the Exercise session:

Definition 1.3. type

Examples 1.4. do it well, types will be very important!! also cover non-realized types.

Definition 1.5. saturated models

Remark 1.6. We don't know the saturated model of the free group!

Fact 1.7. *If there is an automorphism, things have the same type. Converse in saturated models.*

1.2. In the Lecture Class:

Definition 1.8. FORKING

Remark 1.9. Intuition: a forks with B over C, if BC knows a much better, then just C alone. Draw picture for 2-inconsistent (any formula containing a over C. Then forking formula is inside it, by 2-inconsistency we get many infinite disjoint htings, all inside that set as the same type over C).

Example 1.10. algebraic types don't fork. One forking type. Plus some pictures: are they witnessing forking? (semi-disjoint sets)

Fact 1.11. *If T is stable, then the forking independence satisfies the following properties: ...*

Definition 1.12. stable, superstable , omega stable. (via types).

Fact 1.13. *If T is superstable, there are no infinite forking chains. There is a rank, Shelah rank. omega stable has morley rank.*

2. Stable Groups

Definition 2.1. If T is some first order theory and G some group **definable** in T. We say that G is a **stable** (resp. **superstable,** $\omega-$**stable**) **group**, if the theory T is stable (resp. superstable, $\omega-$stable).

Convention 2.2. When we talk about the theory of the free group $\mathbb{F}$, we consider $\mathbb{F}$ as a **pure group**, i.e. in the language of groups $L = \{\cdot, ^{-1}\}$.

Exercise 2.3.
(1) Assume M is an infinite structure in the language of groups, in which the operation $\cdot$ is associative and admits left and right cancellation. Show that, if M is stable, then M already is a group.
HINT: Use that no infinite set can be ordered by a first order formula. Consider the formula $\varphi(x, y) := \exists z(x \cdot z = y)$ together with the sequence $(a^n \mid n \in \mathbb{N})$.
(2) Let $A \subseteq G$ be a definable subset of some stable group G. Show that for any $g \in G$ we have

$$gA = A \text{ if and only if } gA \subseteq A.$$

Lemma 2.4 (Baldwin-Saxl Condition)**.** *Every intersection of uniformly definable subgroups is finite and hence definable*

Exercise 2.5. Centralizers of sets (not necessarily definable) are definable subgroups in stable groups.

Remark 2.6. Chain conditions in superstable (no infinite definable chain with infinite index) and omega-stable (no infinite definable chain of subgroups what so ever)

3. Generics

From now on, we work inside some stable group G.

Definition 3.1. A definable set $A \subseteq G$ is called **left generic**, if finitely many (left-)translates cover the whole group G. Similarly for **right generic**. It is called **bilateral generic**, if finitely many translates of the form gAh cover G.

Exercise 3.2. If some set $A \subseteq G$ is left generic, then its inverse set A^{-1} is right generic.

Lemma 3.3. *For any definable set A, either A is left generic, or its complement $G \setminus A$ is right generic.*

Lemma 3.4. *A definable set A is left generic if and only if it is right generic, if and only if it is bilateral generic. We thus call such a set just* **generic***.*

Definition 3.5. generic type, generic element, generic over some set A.

Corollary 3.6. *Generic types exist. More precisely: Any partial generic type over some parameter set A can be extended to a generic type over A.*

Lemma 3.7. *Let g be generic and algebraic over some element h. Then h also is generic.*

Definition 3.8. φ-connected component, connected component.

Fact 3.9.
- *In stable, $G^0(\varphi)$ is definable of finite index, hence generic.*
- *In omega-stable, connected component definable of finite index, hence generic.*
- *G^0 is a* **characteristic** *subgroup, i.e. invariant under automorphisms. In particular, G^0 is normal in G.*

Maybe exercise: type generic iff stabilizer is G^0.

4. Forking in Stable Groups

Lemma 4.1. *If the type* $\mathrm{tp}(a/B)$ *is generic, for some finite tuple a and an arbitrary subset B of G, then $a \perp B$.*

Lemma 4.2. *If a and b are generic elements, independent over some set C, then so are a and ab.*

Exercise 4.3. In a ω-saturated stable group, any element is product of two generics.