Lecture 1 -Introduction to the free group

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Question 1: *(Tarski problem, 1945) Do free groups of different ranks have the same first order theory?*

Sela answered this question in the affirmative in (around 2003, see also works of Kharlampovich and Miasnikov) - a huge work that analyzes all definable sets in the theory of free groups and proves many other results besides.

Building on this, he also proved

Theorem 0.1: *The common theory of non abelian free groups is stable.*

(It was known it is not superstable - see third lecture). This gives a rare natural example of stable structures (see more on stability in Isabel's lecture).

This sparked a renewed interest in the model theory of the free groups.

1 Some basics about free groups

Let S be a (possibly infinite) set, for each $s \in S$ add a letter $\bar{s}$, and denote $\bar{S}$ the set $\{\bar{s} \mid s \in S\}$. We take the convention that $\bar{\bar{s}} = s$.

Definition 1.1: *A word in S is a finite sequence $u_1 \dots u_n$ of elements of $S \cup \bar{S}$. A word is said to be reduced if for every $i = 1, \dots, n-1$, we have $u_i \neq \bar{u}_i$. If $w = u_1 \dots u_n, w' = v_1 \dots v_m$ are two words, their concatenation is $u_1 \dots u_n v_1 \dots v_m$. We denote $F(S)$ the set of reduced words on S.*

Lemma 1.2: *The set $F(S)$ together with the concatenation reduction operation is a group.*

Definition 1.3: *We call $F(S)$ the free group on S.*

Lemma 1.4: *(Universal property) Let S be a set, and $F(S)$ the free group on S. For any group G, and any choice of elements $(g_s)_{s \in S}$, there is a unique morphism $h : F(S) \to G$ such that $h(s) = g_s$ for all $s \in S$.*

Definition 1.5: *If the morphism $h : F(S) \to G$ is an isomorphism, we say that G is free on $\{g_s \mid s \in S\}$.*

Remark 1.6: *G is free on $\{g_s \mid s \in S\}$ iff $\hat{S}$ generates G and no nontrivial reduced word on S represents the identity element.*

Remark 1.7: *In fact there are three possible ways to define free groups*

1. *The constructive way: given a set S, build $F(S)$ just as we did.*

2. *By the universal property: category theory tells us that in the category of groups there is a unique object which satisfies the property given in Lemma 1.4, we call it the free group.*

3. *The botanical point of view: if you find in nature a group G which admits a generating set S such that no nontrivial reduced products in the elements of S and their inverses is trivial, call it free on S.*

2 Bases and rank

The free group $F(S)$ is of course free on S - the morphism given by Lemma 1.4 is just the identity.

But there are many other sets $T \subseteq F(S)$ such that $F(S)$ is free on T.

Example 2.1: Let $S = \{a, b\}$. Let $T = \{\alpha, \beta\}$ and let $h : F(T) \to F(S)$ be the unique morphism such that $h(\alpha) = a$ and $h(\beta) = ba$.

By universal property of $F(S)$ there is a morphism $f : F(S) \to F(T)$ given by $a \mapsto \alpha$ and $b \mapsto \beta\alpha^{-1}$. Now $g \circ f : F(S) \to F(S)$ fixes both a and b - by uniqueness in the universal property, it must be the identity. Similarly $f \circ g = \mathrm{Id}$ so we see that f, g are in fact isomorphisms.

We conclude that $F(S)$ is free on $\{a, ab\}$.

Definition 2.2: *If G is free on a set $S \subseteq G$ we call S a basis of G. An element is primitive if it is part of some basis.*

Remark 2.3: *By Remark 1.6, S is a basis if it generates and there is no non trivial relation between the elements - this should remind you of the definition of the basis of a vector space.*

Lemma 2.4: *Let S, S' be sets. Then $F(S)$ is isomorphic to $F(S')$ iff $|S| = |S'|$.*

Corollary 2.5: *Any two bases of G have the same cardinality.*

Definition 2.6: *The rank of a free group is the cardinality of a basis.*

Warning. Bases of free groups are not as well-behaved as bases of vector spaces.

1. Not every group admits a basis, only free groups!
2. Not every generating set contains a basis.
3. If S is a subset of G which is free (i.e. no nontrivial reduced word on S represents the identity element) but not generating, it cannot in general be extended to a basis of G.
4. A free group of rank k may have free subgroups of rank $n > k$, indeed of infinite rank!

3 Subgroups

We have the following theorem

Theorem 3.1: *Any subgroup of a free group is free.*

This means that if $H \leqslant \mathbb{F}$, there exists some subset S_H of H such that H is free on S_H.

In general S_H need not extend to a basis of G. But we give the following

Definition 3.2: *Let G be a free group, and H a subgroup of G. We say H is a free factor of G if there exists a basis $S = s_1, \ldots, s_n$ of G such that $H = \langle s_1, \ldots, s_k \rangle$ for some $k \leqslant n$.*

Note that in this case, H is free on $\{s_1, \ldots, s_k\}$ - by definition it generates H, and no nontrivial word on the set S_H represents the identity element.

4 Automorphisms of the free group

We are interested in $\mathrm{Aut}(\mathbb{F}_n)$. Suppose $\mathbb{F}_n$ has basis $a_1, \ldots, a_n$

Example 4.1: All the following are automorphisms:

1. $a_i \mapsto a_{\sigma(i)}$ for all i, for some permutation $\sigma \in S_n$;
2. $a_k \mapsto a_k^{-1}$ and $a_i \mapsto a_i$ for all $i \neq k$;
3. $a_k \mapsto a_k a_l$ and $a_i \mapsto a_i$ for all $i \neq k$.

Theorem 4.2: *(Nielsen) The automorphisms given above generate* $\mathrm{Aut}(\mathbb{F}_n)$.

5 Some facts about free groups

1. Roots are unique
2. Centralizers
3. Commutative transitive

6 Hopf properties

Theorem 6.1: *Free groups of finite rank have the Hopf property: any surjective homomorphism* $\mathbb{F}_k \to \mathbb{F}_k$ *is in fact also injective.*

Proof: Free groups are linear hence residually finite. Residually finite groups are Hopf.

Theorem 6.2: *(Sela) Let* G *be a torsion free hyperbolic group (for example free) and* $A \subseteq G$ *such that* A *is not contained in any proper free factor of* G*. Then any injective morphism* $G \to G$ *fixing* A *pointwise is also surjective.*

7 Free products

Definition 7.1: *Let* A, B *be two groups. A word in* A, B *is an alternating sequence* $(a_1)b_1a_2 \ldots b_{l-1}a_l(b_l)$ *of elements* $a_i \in A$*,* $b_i \in B$*, it is reduced if the elements are non trivial. The set* $A * B$ *of reduced words together with the operation of concatenation reduction (replace 2 consecutive elements in the same group by their product, erase if trivial) is a group called the free product of* A *and* B*.*

Proposition 7.2: *If* G *is a group with subgroups* A, B *such that*

1. $A \cup B$ *generates* G*;*
2. *no reduced alternating product of non trivial elements of* A, B *is trivial;*

then G *is isomorphic to* $A * B$ *via the identity on* A, B *and we write in fact* $G = A * B$*.*

Definition 7.3: *This is called a free factor decomposition for* G*. The subgroups* A *and* B *are called free factors of* G*.*

If G *has no nontrivial subgroups* A, B *such that* $G = A * B$ *we say* G *is freely indecomposable.*

Proposition 7.4: *If* $G = A * B$*, then* $\mathrm{rk}(G) = \mathrm{rk}(A) + \mathrm{rk}(B)$ *(where* $\mathrm{rk}(A)$ *is the smallest size of a generating set for* A*).*

Theorem 7.5: *(Grushko) Let* G *be a finitely presented group. Then there exist* $k \in \mathbb{N}$ *and* $G_1, \ldots, G_r$ *subgroups of* G *such that*

1. $G = G_1 * \ldots * G_r * \mathbb{F}_k$*;*
2. G_i *is freely indecomposable for all* i*.*

Moreover, if $G = H_1 * \ldots * H_s * \mathbb{F}_l$ *is another decomposition which satisfies the same properties, then* $r = s$*,* $k = l$*, and up to permutation each* H_i *is a conjugate of* G_i*.*

Theorem 7.6: *(Kurosh) Let* $G = A * B$ *and* $H \leqslant G$*. Then*

$$H = \ast_{A^g \text{ conj to } A}(A^g \cap H) * \ast *_{B^g \text{ conj of } A}(B^g \cap H) * \mathbb{F}_k$$

for some $k \in \mathbb{N}$*.*

Corollary 7.7: *If* H *and* K *are both free factors of* G *then* $H \cap K$ *is also a free factor of* G*.*

Proof. We have $G = H*H'$, so $K = (K \cap H)*K_0$ by Kurosh. But $G = K*K'$ thus $G = (K \cap H)*K_0*K'$ which proves the result. □

Corollary 7.8: *If $M \subset G$, the intersection of all the free factors of G containing M is a free factor of G, we call it the minimal free factor containing M.*

Proof. When intersect free factors, either the rank goes down by proposition 7.4 (this can only happen finitely many times), or the intersection is equal to one of the free factors. □

Proposition 7.9: *Suppose $M \subseteq G$. The minimal free factor G_M containing M is characteristic, i.e. for any automorphism $f : G \to G$ such that $f(M) = M$, we have $f(G_M) = G_M$.*

Proof. f sends any free factor containing M to another free factor containing M so the intersection is preserved. □