Lecture 11 - Forking in the free group - Part 2

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The aim of the lecture is to describe the forking independence relation in a free group $\mathbb{F}$ over a parameter set A relative to which $\mathbb{F}$ is freely indecomposable. The result is given in terms of the relative cyclic JSJ decomposition Λ of $\mathbb{F}$ relative to A, and of the minimal subgraphs of the subgroups $\langle A, b\rangle_{\mathbb{F}}$ and $\langle A, c\rangle_{\mathbb{F}}$ in Λ so we start by giving

Definition 0.1: *Let G be a group and let H be a finitely generated subgroup. Let Λ be a graph of groups decomposition for G, denote by T the corresponding tree (given by Bass-Serre theory) on which G acts and by $p : T \to \Lambda$ the quotient map. The minimal subgraph Λ_H^{min} of H in Λ is $p(T_H^{min})$, where T_H^{min} is the minimal subtree of T invariant by H.*

More concretely (but less precisely): if we choose a spanning tree for Λ we get a presentation for G whose generators are elements of the vertex groups and Bass-Serre elements corresponding to edges outside the spanning tree. To each element $g \in G$ one can associate a minimal closed path. To do so, write g as a product in the generators (with usual rules about not belonging to edge groups if it can be avoided), and draw the path described by the product as follows:

1. if the product starts by u, start at the vertex to whose group u belongs or start at the beginning of the edge corresponding to the Bass-Serre element u and cross that edge;
2. at each new generator u appearing in the product either (i) go via the spanning tree to the vertex whose group contains u or (ii) go via the spanning tree to the start of the edge corresponding to the Bass-Serre element u and cross said edge;
3. at the end of the product return to starting point via the spanning tree.

The minimal subgraph Λ_H^{min} is the union of all the cycles associated to elements of H.

Theorem 0.2: *Let b, c be tuples in $\mathbb{F}$ and $A \subseteq \mathbb{F}$ be such that $\mathbb{F}$ is freely indecomposable relative to A. Let Λ be the cyclic JSJ decomposition of $\mathbb{F}$ relative to A, and denote by Λ_{Ab}^{min} and Λ_{Ac}^{min} the minimal subgraphs of the subgroups $\langle A, b\rangle_{\mathbb{F}}$ and $\langle A, c\rangle_{\mathbb{F}}$ in Λ.*

Then b and c are independent over A if and only if Λ_{Ab}^{min} and Λ_{Ac}^{min} intersect at most in a disjoint union of (envelopes of) rigid vertices.

Recall def of minimal subgraph.

Example 0.3:

1. Two edges adjacent in A, b and c in extremities.
2. Y graph with non Z vertex in the middle.
3. Bigon with rigid, conjugate elements.
4. Bigon with surface, non conjugate elements.

1 Some reminders

Definition 1.1: *$b, c \in M$ and $A \subseteq M$. Say b and c* ***fork over*** *A if there exists*

- *a set X definable over Ac containing b;*

- *a sequence* $\theta_n \in \mathrm{Aut}_A(\hat{M})$ *(where* $\hat{M}$ *is some elementary extension of* M*);*
- *an intyeger* k

such that the sets $\theta_n(X)$ *are* k*-wise disjoint.*

Note that any set X which is definable over Ac and contains b will contain $\mathrm{Aut}_{Ac}(M) \cdot b$ (the orbit of b under $\mathrm{Aut}_{Ac}(M)$).

Theorem 1.2: *Let* $A \subseteq \mathbb{F}$ *be such that* $\mathbb{F}$ *is freely indecomposable relative to* A*. Then the orbit of any tuple* $g \in \mathbb{F}$ *under* $\mathrm{Aut}_A(\mathbb{F})$ *is definable over* A*.*

2 Modular groups and minimal subgraphs

Let G be a torsion free hyperbolic group, and A a non abelian (for simplicity) subgroup not contained in any proper free factor of G. Let Λ be the cyclic JSJ decomposition of G relative to A.

Definition 2.1: *An* ***elementary automorphism*** *associated to* Λ *is either a Dehn twist corresponding to an edge of* Λ *or a surface type automorphism associated to a surface vertex of* Λ*.*

Recall that the modular group $\mathrm{Mod}_A(G)$ is the subgroup of $\mathrm{Aut}_A(G)$ generated by all Dehn twists fixing H associated to splittings of G as $G = U *_Z$ or $G = U *_Z V$ with Z infinite cyclic and $A \leq A$. Because the JSJ encodes all such splittings, we can in fact show that $\mathrm{Mod}_A(G)$ is generated by elementary automorphisms of Λ, and more precisely

Proposition 2.2: *Any element of* $\mathrm{Mod}_A(\mathbb{F})$ *can be written as a product* $\mathrm{Conj}(\gamma) \circ \tau_1 \circ \ldots \circ \tau_m$ *where each* τ_i *is an elementary automorphism of* Λ *on a different support, and the order of supports can be permuted.*

The essential property of minimal subgraphs we will use:

Lemma 2.3: *Let* $g \in \mathbb{F}$*, and let* $\tau \in \mathrm{Mod}_A(\mathbb{F})$ *be a Dehn twist associated to an edge of* $\Lambda - \Lambda^{min}_{Ag}$ *or a surface type automorphism associated to a surface type vertex of* $\Lambda - \Lambda^{min}_{Ag}$*. Then* $\tau(g) = g$*.*

3 Proving independence

Suppose the conditions hold. We want to compare how the definable structure over A and that over Ac capture b. Note that $\mathrm{Aut}_A(\mathbb{F}) \cdot b$ and $\mathrm{Aut}_{Ac}(\mathbb{F}) \cdot b$ are definable, so they are the smallest definable sets of each structure which contain b - we want to compare them.

We will show that under the assumption of the theorem, they are almost the same!

Lemma 3.1: *If the minimal subgraphs satisfy the conditions in the Theorem, we have* $\mathrm{Mod}_{Ac}(\mathbb{F}) \cdot b = \mathrm{Mod}_A(\mathbb{F}) \cdot b$*, in particular* $\mathrm{Mod}_{Ac}(\mathbb{F}) \cdot b$ *is invariant under* $\mathrm{Mod}_A(\mathbb{F})$*.*

Proof. The modular group $\mathrm{Mod}_A(\mathbb{F})$ is generated by Dehn twists and surface type automorphisms. If the minimal subgraphs do not intersect in an edge or a surface group, any image of b obtained by applying an element of $\mathrm{Mod}_A(\mathbb{F})$ can be obtained by an element which fixes c (the Dehn twists or surface type automorphisms which do not fix c do nothing to b). □

We deduce

Proposition 3.2: *There exists sets* $Z_1, \ldots Z_m$ *such that any translate of the set* $\mathrm{Aut}_{Ac}(\mathbb{F}) \cdot b$ *by an element of* $\mathrm{Aut}_A(\mathbb{F})$ *contains one of the* Z_j*.*

Proof. Take $Z_1 = \mathrm{Mod}_{Ac}(\mathbb{F}) \cdot b$. Since $\mathrm{Mod}_A(\mathbb{F})$ has finite index in $\mathrm{Aut}_A(\mathbb{F})$ and Z_1 is preserved by $\mathrm{Mod}_A(\mathbb{F})$, this implies that Z_1 has finitely many translates $Z_1, \ldots Z_m$ under $\mathrm{Aut}_A(\mathbb{F})$. Since $\mathrm{Aut}_{Ac}(\mathbb{F}) \cdot b$ contains Z_1, each of its translates by an element of $\mathrm{Aut}_A(\mathbb{F})$ must contain one of the Z_j. □

This implies that for any set X definable over Ac containing b, for any sequence $\theta_n \in \mathrm{Aut}_A(\mathbb{F})$, an infinite number of translates $\theta_n(X)$ contain the same Z_j, hence the $\theta_n(X)$ are not k-wise disjoint for any k.

However we need to prove that this holds for any sequence of automorphisms θ_n of any extension $\hat{\mathbb{F}}$ of $\mathbb{F}$, which is stronger! The trick is to write this as a sentence true on $\mathbb{F}$.

Remark 3.3: *We have in fact that for any set X definable over Ac containing b, any k, any $(k-1)m+1$ translates of X by elements of $\mathrm{Aut}_A(\mathbb{F})$ are not k-wise disjoint.*

We can express this as a first-order sentence as follows: let $\psi_c(y, A)$ be a formula defining the orbit of c under $\mathrm{Aut}_A(\mathbb{F})$, and let $\phi(x, c, A)$ be the formula defining X. If $\theta \in \mathrm{Aut}_A(\mathbb{F})$ then $\theta(X)$ is defined by $\phi(x, \theta(c), A)$.

We have

$$\mathbb{F} \models \forall y_1, \ldots, y_{(k-1)m+1} \left[\bigwedge_{i=1}^{(k-1)m+1} \psi_c(y_i, A) \to \bigvee_{1 \leq j_1 < \ldots < j_k \leq (k-1)m+1} \exists x \bigwedge_{i=1}^{k} \phi(x, y_{j_i}, A) \right]$$

thus this holds in any elementary extension $\hat{\mathbb{F}}$ of $\mathbb{F}$, which proves the result.

4 Proving forking

Suppose the minimal subgraphs Λ_{Ab}^{min} and Λ_{Ac}^{min} intersect in an edge or in a surface type vertex group, and denote by τ an elementary automorphism supported by this edge or vertex. The idea is to show that there is an element in $\mathrm{acl}^e q(A, b)$ whose orbit under $\mathrm{Mod}_{Ac}(\mathbb{F})$ has infinitely many disjoint translates under τ^l.

In the Y example, we have in fact that $\mathrm{acl}(Ab)$ and $\mathrm{acl}(Ac)$ contain the central vertex group, which is not in $\mathrm{acl}(A)$ - this is enough to prove that b and c fork.

More generally, if the minimal subgraphs Λ_{Ab}^{min} and Λ_{Ac}^{min} both contain an edge with end vertex groups U, V which are not cyclic, then without loss of generality the tuple (u, v) (where $U = \langle u \rangle$ and $V = \langle v \rangle$) lies in $\mathrm{acl}^{eq}(Ab)$. We then show that the orbit of (u, v) under $\mathrm{Mod}_{Ac}(\mathbb{F})$ lies in a single conjugacy class. But if we apply powers of the Dehn twist τ supported by the edge e, we can show that we get tuples which lie in infinitely many conjugacy classes. This implies that (u, v) forks with c over A, hence so does b.

Finally, if the minimal subgraphs Λ_{Ab}^{min} and Λ_{Ac}^{min} both contain a surface vertex, there are elements β and γ of the surface group which correspond to simple closed curves on the surface and lie in $\mathrm{acl}(Ab)$ and $\mathrm{acl}(Ac)$ respectively. It is then possible to show (using pseudo-Anosov automorphisms...) that β and γ fork over A, which implies the result.