MODEL THEORY OF THE FREE GROUP

EXERCISE CLASS 4

1. Dehn-Twists Revisited

Fact 1.1. *Assume you have a decomposition of G as $G = A *_C B$ and $g = (a_1)b_1a_2 \dots a_l(b_l)$ such that*

$$a_i \in A \setminus C \text{ and } b_i \in B \setminus C.$$

Then $g \neq 1$.

Fact 1.2 (Britton's Lemma)**.** *Assume $G = A*_C = \langle A, t \mid tc_1t^{-1} = c_2 \rangle$. If $g = a_1t^{k_1}a_2t^{k_2} \dots a_lt^{k_l}a_{l+1} = 1$ with $k_i \neq 0$, then there exists some i such that either*

$$k_{i-1} > 0 > k_i \text{ and } a_i \in C_1$$

or

$$k_{i-1} < 0 < k_i \text{ and } a_i \in C_2.$$

Exercise 1. Let $G = A *_C B$.

(i) Let $\gamma = (\alpha_1)\beta_1\alpha_2 \dots \alpha_k(\beta_k)$ with $\alpha_i \in A \setminus C$ and $\beta_i \in B \setminus C$. Suppose $\gamma \in Cen_G(C)$ the centralizer of C in G. Show that $\alpha_i \in N_A(C)$ and $\beta_i \in N_B(C)$ for all i.

(ii) Define $\tau_\gamma : G \to G$ by $\tau_\gamma(a) = a$ for all $a \in A$ and $\tau_\gamma(b) = \gamma b \gamma^{-1}$ for all $b \in B$. Show that this is a homomorphism.

(iii) Show that τ_γ is an isomorphism if and only if $\gamma = \alpha_1\beta_1$.

Fact 1.3. *Assume $A *_C B$ is a free group and $C = \langle c \rangle$ is cyclic. Then c is primitive in one of the factors A or B.*

Exercise 2. Assume that $F = A *_C B$ is a free group. Show:

(i) If there are non-trivial Dehn-twists corresponding to the above decomposition, then C is cyclic.

(ii) If C is non-trivial, then any Dehn-twist corresponding to the above decomposition is an automorphism of F.

2. Homogeneity

Recall the idea of proving Homogeneity for the free group: For elements $u, v \in \mathbb{F}$ having the type we show:

(1) There is a homomorphism θ sending u to v.
(2) We can pick θ to be injective.
(3) If there are monomorphisms $\theta, \theta' : \mathbb{F} \to \mathbb{F}$ such that $\theta(u) = v$ and $\theta'(v) = u$, then u and v are in the same orbit under $\mathrm{Aut}(\mathbb{F})$.

We will repeat the proof of homogeneity in this class for some basic cases.

Exercise 3. Show that there is a homomorphism θ from $\mathbb{F}$ to $\mathbb{F}$ sending u to v.

Exercise 4. Let u be any element of the free group. Show that there is a minimal free factor F_u of $\mathbb{F}$ which contains u.

Exercise 5. Show in the cases

(i) $Mod_u(F_u) = \{e\}$ and
(ii) $Mod_u(F_u) = \{\tau_\gamma \mid \gamma \in Cen(C)\}$, where the τ_γ are the Dehn-twists corresponding to the decomposition $F_u = A *_C B$ with $C \cong \mathbb{Z}$,

that the morphism θ can be chosen to be injective.
Proceed as follows:

- Recall Proposition 1.4 from lecture 4: If F is freely indecomposable over u and $\theta : \langle u \rangle \to F$ a homomorphism. Then there are finitely many proper quotients $Q_1, \dots Q_k$ of F such that for any morphism θ' from F to F which extends θ and is not injective, there is $\sigma \in Mod_u(F)$ such that $\theta' \circ \sigma$ factors through one of the Q_i. Which F should we consider here?
- As the quotients are proper, there are non-trivial elements g_i in the kernel of the projections $\eta_i : F \to Q_i$. Can you modify the construction of θ in Exercise 3, so to ensure that $\theta(g_i) \neq e$ for all i?
- Conclude from there that in the Case (i) we can pick θ to be injective.
- For Case (ii) we have to ensure that $\theta(\sigma(g_i)) \neq e$ for all i and $\sigma \in Mod_u(F)$. First describe how $Mod_u(F)$ looks like. Then try to mimic the solution from Case (i).

Exercise 6. Assume there are monomorphisms $\theta_u : F_u \to \mathbb{F}$ and $\theta_v : F_v \to \mathbb{F}$ such that $\theta_u(u) = v$ and $\theta_v(v) = u$. Show that u and v are in the same orbit under $\mathrm{Aut}(\mathbb{F})$. Proceed as follows:

(i) Show that $\theta_u(F_u) \leq F_v$.
(ii) Deduce that $\theta_{u|F_u} : F_u \overset{\cong}{\to} F_v$ is an isomorphism.
(Hint: Remember that free groups are relatively co-Hopfian)
(iii) Conclude that there is an automorphism of $\mathbb{F}$ sending u to v.